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Abstract.   The massive N-flavour Schwinger model is analysed by the bosonization method. The problem is reduced to the quantum mechanics of N degrees of freedom in which the potential needs to be self-consistently determined by its ground-state wavefunction and spectrum with given values of the θ parameter, fermion masses, and temperature. Boson masses and fermion chiral condensates are evaluated. In the $N = 1$ model the anomalous behaviour is found at $\theta \sim \pi$ and $m/\mu \sim 0.4$. In the $N = 3$ model an asymmetry in fermion masses $(m_1 < m_2 \ll m_3)$ removes the singularity at $\theta = \pi$ and $T = 0$. The chiral condensates at $\theta = 0$ are insensitive to the asymmetry in fermion masses, but are significantly sensitive at $\theta = \pi$. The resultant picture is similar to that obtained in QCD by the chiral Lagrangian method.

1. Introduction

Two-dimensional quantum electrodynamics (QED_2) [1–4] with massive N-flavour fermions resembles four-dimensional quantum chromodynamics (QCD) in many respects [5]. In both theories nonvanishing chiral condensates are dynamically generated. Fractionally charged test particles are confined in QED_2, whereas quarks, or coloured objects, are confined in QCD. The dynamical chiral symmetry breaking and confinement are not independent phenomena in QED_2, however. There would be no confinement if there were no chiral condensates [6, 7].

It has also been recognized that QED_2 describes spin systems in nature [8]. A spin $\frac{1}{2}$ anti-ferromagnetic spin chain is equivalent to a two-flavour massless Schwinger model in a uniform background charge density. An n-leg spin ladder systems is equivalent to a coupled set of n Schwinger models. This equivalence has been successfully employed to account for the gap generation in spin ladder systems [9].

QED_2 has been analysed by various methods. On the analytic side it has been investigated using perturbation theory, the path-integral method, and the bosonization method. In a series of papers we have shown how to evaluate chiral condensates and boson mass spectrum for arbitrary values of the θ parameter, fermion masses (m), and temperature (T) by bosonization [7, 10–12]. The mass perturbation theory has been formulated for the one-flavour model [13, 14].

Investigation in the light-cone quantization method has been pushed forward both on the analytic and numerical sides [15–20]. The bound-state spectrum has been evaluated in the entire range of fermion mass at $\theta = 0$ and $T = 0$. Subtleties in the chiral condensate in this formalism has been noted [17].

There has emerged a renewed interest in QED_2 in the lattice gauge theory approach as well [21–23]. Recently extensive simulations have been carried out for the $N = 1$ and

$N = 2$ models. Chiral condensates in the $N = 1$ model were evaluated at $\theta = 0$ and $T = 0$ up to $m/e < 1$. The boson mass in the $N = 2$ model was evaluated for $m/e < 0.5$. After subtracting condensates in free theory ($e = 0$), depending on the regularization methods employed, one finds a modest agreement between the bosonization and lattice results [23].

In this paper we further exploit the bosonization method to investigate the dependence of chiral condensates and boson mass spectrum on the θ parameter, fermion masses, and temperature. The advantage of our method lies in the ability to evaluate physical quantities for arbitrary values of the θ parameter and temperature. The current method, however, involves an approximation which is not valid for large fermion masses. Improvement is necessary in this direction.

The Lagrangian of the model is given by

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_a \{\bar{\psi}_a \gamma^\mu (\mathrm{i}\partial_\mu - e_a A_\mu)\psi_a - m_a \bar{\psi}_R^a \psi_L^a - m_a^* \bar{\psi}_L^a \psi_R^a\} \tag{1.1}$$

where $\psi_L^a = \tfrac{1}{2}(1 - \gamma^5)\psi_a$ and $\psi_R^a = \tfrac{1}{2}(1 + \gamma^5)\psi_a$. Each field carries a charge e_a and mass m_a. We analyse the model on a circle (S^1) with a circumference L. The boundary conditions are specified by

$$A_\mu(t, x + L) = A_\mu(t, x)$$
$$\psi_a(t, x + L) = -\mathrm{e}^{2\pi\mathrm{i}\alpha_a}\psi_a(t, x). \tag{1.2}$$

It is important to recognize that from the analysis on S^1 one can extract physics at finite temperature T defined on a line R^1. In the Matsubara formalism of finite temperature field theory, boson and fermion fields obey a periodic or anti-periodic boundary condition in the imaginary time axis, respectively;

$$A_\mu\left(\tau + \frac{1}{T}, x\right) = A_\mu(\tau, x)$$
$$\psi_a\left(\tau + \frac{1}{T}, x\right) = -\psi_a(\tau, x). \tag{1.3}$$

Hence, if one, in a theory defined on S^1 with the boundary conditions $\alpha_a = 0$ in (1.2), analytically continues t from the real axis to the imaginary axis, and then interchanges (or relabels) it and x, one arrives at a theory which is exactly the same as the theory defined on R^1 at $T = L^{-1}$. This is a powerful equivalence. One can evaluate chiral condensates, Polyakov loops, and various correlators at $T \neq 0$ with the aid of this correspondence.

This paper is organized as follows. In section 2 the bosonized Hamiltonian is derived on S^1. In section 3 the θ vacuum is introduced and the equation satisfied by its wavefunction is derived. In section 4 we show how the boson mass spectrum and chiral condensates are evaluated. Sections 3 and 4 together form the basis of our formulation. It defines the generalized Hartree–Fock approximation. The rest of the paper is devoted to applying the generalized Hartree–Fock equation to various models to evaluate the boson spectrum and chiral condensates. The case of massless fermions is discussed in section 5. A useful truncated formula is derived in section 6. The detailed analysis of the massive one-flavour model is given in section 7. The multiflavour model with degenerate fermion masses is analysed in section 8. The case of general masses in the $N = 2$ and $N = 3$ models is investigated in section 9. A brief summary is given in section 10. Three appendices collect useful formulae.

2. The bosonized Hamiltonian

Our basic tool is the bosonization method, with which we shall reduce the model (1.1) to a quantum mechanical system of finite degrees of freedom. The bosonization method has been developed in many-body theory [24], and in field theories on a line R^1 [25, 3, 26]. The method has been elaborated on a manifold S^1 in the context of string theory [27].

The bosonization on S^1 is particularly unamabiguous, sustaining the absolute rigour. QED_2 on S^1 was first studied by Nakawaki [28] and has been developed by many authors [29–31]. It was simplified in [30], which we follow here.

In this section we present a brief review of the method, applying it to the system (1.1). Although the essence is well known in the literature, subtle factors associated with the multi-flavour nature and implementation of arbitrary boundary conditions are worth spelling out. With a clever choice of Klein factors the Hamiltonian of the N-flavour Schwinger model is transformed into a surprisingly simple bosonized form.

We note that the N-flavour Schwinger model has been analysed in [26, 10, 11, 32]. The model at finite temperature, which is equivalent to the model on S^1, has been analysed in [33]. The model on other manifolds have also been investigated in [34]. The conformal field theory approach to QED_2 has been proposed by Itoi and Mukaida [35], which has many features in common with our bosonization approach. Correlators of various physical quantities have been discussed in [36].

Bosonization of an arbitrary number of fermions on a circle $(0 < x < L)$ obeying boundary conditions

$$\psi_a(t, x + L) = -e^{2\pi i \alpha_a} \psi_a(t, x) \qquad a = 1 \sim N \tag{2.1}$$

is first carried out in the interaction picture defined by free massless fermions: $i\gamma \partial \psi = 0$ [26, 4]. We introduce bosonic variables:

$$[q_\pm^a, p_\pm^b] = i\delta^{ab} \qquad [a_{\pm,n}^a, a_{\pm,m}^{b,\dagger}] = \delta^{ab}\delta_{nm}$$

all other commutators $= 0$

$$\phi_\pm^a(t, x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} \{a_{\pm,n}^a e^{-2\pi i n(t \pm x)/L} + \text{h.c.}\}. \tag{2.2}$$

In terms of these variables $\psi_a^t = (\psi_+^a, \psi_-^a)$ can be expressed as [27]

$$\psi_\pm^a(t, x) = \frac{1}{\sqrt{L}} = C_\pm^a e^{\pm i\{q_\pm^a + 2\pi p_\pm^a(t \pm x)/L\}} : e^{\pm i\sqrt{4\pi}\phi_\pm^a(t,x)} :$$

$$= \pm \frac{1}{\sqrt{L}} e^{\pm i\{q_\pm^a + 2\pi p_\pm^a(t \pm x)/L\}} C_\pm^a : e^{\pm i\sqrt{4\pi}\phi_\pm^a(t,x)} : \tag{2.3}$$

where the Klein factors are given by [10]

$$C_+^a = \exp\left\{ i\pi \sum_{b=1}^{a-1}(p_+^b + p_-^b - 2\alpha_b) \right\}$$

$$C_-^a = \exp\left\{ i\pi \sum_{b=1}^{a}(p_+^b - p_-^b) \right\}. \tag{2.4}$$

This choice of Klein factors turns out to be very convenient. They satisfy

$$\begin{bmatrix} C_+^a \\ C_+^{a\dagger} \end{bmatrix} \psi_\pm^b(x) = \text{sign}(b \geq a)\psi_\pm^b(x) \begin{bmatrix} C_+^a \\ C_+^{a\dagger} \end{bmatrix}$$

$$\begin{bmatrix} C_-^a \\ C_-^{a\dagger} \end{bmatrix} \psi_\pm^b(x) = \text{sign}(b > a)\psi_\pm^b(x) \begin{bmatrix} C_-^a \\ C_-^{a\dagger} \end{bmatrix} \tag{2.5}$$

where sign(A) is defined to be $+1$ (-1), when A is true (false). By construction $(\partial_t \mp \partial_x)\psi_\pm^a(t, x) = 0$, i.e. ψ_a satisfies a free massless Dirac equation: $(\gamma^0 \partial_0 + \gamma^1 \partial_1)\psi^a = 0$ where $\gamma^\mu = (\sigma_1, i\sigma_2)$ and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$.

Under a translation along the circle

$$\psi_\pm^a(t, x+L) = -e^{2\pi i p_\pm^a}\psi_\pm^a(t, x) = -\psi_\pm^a(t, x)e^{2\pi i p_\pm^a} \tag{2.6}$$

so that the boundary condition (1.2) is ensured by a physical state condition

$$e^{2\pi i p_\pm^a}|\text{ phys }\rangle = e^{2\pi i \alpha_a}|\text{ phys }\rangle. \tag{2.7}$$

Further conditions can be consistently imposed on physical states such that the Klein factors act in physical space as a semi-identity operator: $C_\pm^a|\text{phys}\rangle = (+ \text{ or } -)|\text{phys}\rangle$. We shall see below that the Hamiltonian commutes with $p_+^a - p_-^a$.

The fields $\{\psi_\pm^a(x)\}$ satisfies desired equal-time anti-commutation relations. With the aid of (2.5), (B.1) and (B.2), it is straightforward to show

$$\{\psi_\alpha^a(x), \psi_\beta^b(y)^\dagger\} = \delta^{ab}\delta_{\alpha\beta}e^{i\pi(x-y)/L} \cdot e^{2\pi i p_a^a(x-y)/L}\delta_L(x-y) \tag{2.8}$$
$$\text{others} = 0$$

where $\alpha, \beta = +$ or $-$. Notice that the extra phase factors in (2.8) manifest the translation property (2.6). The bosonization in the interaction picture is defined by (2.3) and (2.7).

In applying the bosonization method to the model (1.1), it is most convenient to take the Coulomb gauge

$$A_1(t, x) = b(t)$$

$$A_0(t, x) = -\int_0^L \mathrm{d}y\, G(x-y)j_{\text{EM}}^0(t, y) \qquad j_{\text{EM}}^0 = \sum_a e_a\psi_a^\dagger\psi_a \tag{2.9}$$

$$G(x+L) = G(x) \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x) = \delta_L(x) - \frac{1}{L}$$

in which the zero mode $b(t)$ of $A_1(t, x)$ is the only physical degree of freedom associated with the gauge fields. The Hamiltonian is given by

$$H = \frac{1}{2L}P_b^2 + \int_0^L \mathrm{d}x \sum_a \{\bar\psi_a\gamma^1(-i\partial_1 + e_ab)\psi_a + m_a\bar\psi_{aL}\psi_{aR} + m_a^*\bar\psi_{aR}\psi_{aL}\}$$

$$-\frac{1}{2}\int_0^L \mathrm{d}x\, \mathrm{d}y\, j_{\text{EM}}^0(x)G(x-y)j_{\text{EM}}^0(y). \tag{2.10}$$

Here P_b is the momentum conjugate to b: $P_b = L\dot b$. The antisymmetrization of fermion operators is understood.

At all stages of bosonization, the gauge invariance must be maintained. Due caution is necessary in bosonizing a product of two field operators at the same point as it has to be regularized. At equal time the bosonization formula (2.3) leads to

$$e^{-ie_ab(y-x)}\frac{1}{2}[\psi_\pm^a(y)^\dagger, \psi_\pm^a(x)] = \mp\frac{1}{2\pi i}\left\{\frac{P}{x-y} + i\left(\frac{2\pi p_\pm^a}{L} + e_ab \pm \sqrt{4\pi}\partial_y\phi_\pm^a\right) + (x-y)\right.$$

$$\left[\frac{\pi^2}{6L^2} - \frac{1}{2} : \left(\frac{2\pi p_\pm^a}{L} + e_ab \pm \sqrt{4\pi}\partial_y\phi_\pm^a\right)^2 : \pm\frac{i\sqrt{4\pi}}{2}\frac{\partial^2\phi_\pm^a}{\partial y^2}\right] + \cdots \left.\right\}. \tag{2.11}$$

Gauge-invariant regularization amounts to dropping the $P/(x-y)$ term.

Hence we have

$$\frac{1}{2}[\psi_\pm^{a\dagger}, \psi_\pm^a] = \mp\frac{1}{2\pi}\left(\frac{2\pi p_\pm^a}{L} + e_a b \pm \sqrt{4\pi}\partial_x\phi_\pm^a\right)$$

$$\pm\frac{1}{2}[\psi_\pm^{a\dagger}, (\mathrm{i}\partial_x - e_a b)\psi_\pm^a] = -\frac{\pi}{12L^2}$$
$$+ \frac{1}{4\pi} : \left(\frac{2\pi p_\pm^a}{L} + e_a b \pm \sqrt{4\pi}\partial_y\phi_\pm^a\right)^2 : \mp\frac{\mathrm{i}}{\sqrt{4\pi}}\partial_x^2\phi_\pm^a \tag{2.12}$$

$$\pm\frac{1}{2}[(-\mathrm{i}\partial_x - e_a b)\psi_\pm^{a\dagger}, \psi_\pm^a] = -\frac{\pi}{12L^2}$$
$$+ \frac{1}{4\pi} : \left(\frac{2\pi p_\pm^a}{L} + e_a b \pm \sqrt{4\pi}\partial_y\phi_\pm^a\right)^2 : \pm\frac{\mathrm{i}}{\sqrt{4\pi}}\partial_x^2\phi_\pm^a.$$

In particular, currents are given by

$$j_a^0 = +\frac{1}{2}[\psi_+^{a\dagger}, \psi_+^a] + \frac{1}{2}[\psi_-^{a\dagger}, \psi_-^a] = \frac{-p_+^a + p_-^a}{L} - \frac{1}{\sqrt{\pi}}\partial_x\phi_a$$
$$j_a^1 = -\frac{1}{2}[\psi_+^{a\dagger}, \psi_+^a] + \frac{1}{2}[\psi_-^{a\dagger}, \psi_-^a] = \frac{+p_+^a + p_-^a}{L} + e_a\frac{b}{\pi} + \frac{1}{\sqrt{\pi}}\partial_t\phi_a \tag{2.13}$$

where $\phi_a = \phi_+^a + \phi_-^a$. In terms of $\tilde{\phi}_a = \tilde{\phi}_{a+} + \tilde{\phi}_{a-}$ where $\tilde{\phi}_{a\pm} = (4\pi)^{-1/2}[q_\pm^a + 2\pi p_\pm^a(t \pm x)/L] + \phi_\pm^a$, the current takes a simpler form

$$j_a^\mu = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\tilde{\phi}_a + \delta^{\mu 1}\frac{e_a b}{\pi}. \tag{2.14}$$

In the following discussions, however, we shall find that treating the zero-mode and oscillatory-mode parts separately is more convenient.

The kinetic energy term is transformed to

$$-\mathrm{i}\bar{\psi}_a\gamma^1 D_1\psi_a \equiv \frac{\mathrm{i}}{4}\{[\psi_+^{a\dagger}, D_1\psi_+^a] - [D_1\psi_+^{a\dagger}, \psi_+^a] - [\psi_-^{a\dagger}, D_1\psi_-^a] + [D_1\psi_-^{a\dagger}, \psi_-^a]\}$$

$$= -\frac{\pi}{6L^2} + \frac{1}{4\pi} : \left(\frac{2\pi p_+^a}{L} + e_a b + \sqrt{4\pi}\partial_x\phi_+^a\right)^2 :$$
$$+ \frac{1}{4\pi} : \left(\frac{2\pi p_-^a}{L} + e_a b - \sqrt{4\pi}\partial_x\phi_-^a\right)^2 : . \tag{2.15}$$

Putting all things together, we find

$$H = H_0 + H_\phi + H_{\text{mass}}$$

$$H_0 = -\frac{\pi N}{6L} + \frac{P_b^2}{2L} + \frac{\pi}{2L}\sum_{a=1}^N\left\{(p_+^a - p_-^a)^2 + \left(p_+^a + p_-^a + \frac{e_a b L}{\pi}\right)^2\right\}$$

$$H_\phi = \int_0^L \mathrm{d}x\, \frac{1}{2} : \left\{\sum_{a=1}^N(\dot{\phi}_a^2 + \phi_a'^2) + \mu^2\bar{\phi}^2\right\} : \tag{2.16}$$

$$\mu^2 = \frac{1}{\pi}\sum_a e_a^2 \equiv \frac{\bar{e}^2}{\pi} \qquad \bar{\phi} = \sum_a \frac{e_a}{\bar{e}}\phi_a$$

where H_{mass} represents the fermion mass term. It is convenient to express the Hamiltonian in terms of

$$
\begin{aligned}
q_a &= q_+^a + q_-^a & p_a &= \tfrac{1}{2}(p_+^a + p_-^a) \\
\tilde{q}_a &= \tfrac{1}{2}(q_+^a - q_-^a) & \tilde{p}_a &= p_+^a - p_-^a \\
[q_a, p_b] &= [\tilde{q}_a, \tilde{p}_b] = i\delta_{ab} & \text{all others} &= 0.
\end{aligned}
\tag{2.17}
$$

H_0 becomes

$$
H_0 = -\frac{\pi N}{6L} + \frac{P_b^2}{2L} + \frac{1}{2\pi L}\sum_{a=1}^{N}\{(e_a bL + 2\pi p_a)^2 + \pi^2 \tilde{p}_a^2\}.
\tag{2.18}
$$

A few important conclusions can be drawn in the massless fermion case ($H_{\mathrm{mass}} = 0$). The zero-mode part H_0 and oscillatory part H_ϕ decouple from each other. Each part is bilinear in operators so that the Hamiltonian is solvable. The oscillatory part consists of one massive boson ($\bar{\phi}$) with a mass μ and $N - 1$ massless bosons. The zero-mode part must be solved with the physical state condition

$$
\begin{aligned}
e^{2\pi i p_\pm^a}| \text{ phys } \rangle &= e^{2\pi i\alpha_a}| \text{ phys } \rangle \\
Q^{\mathrm{EM}}| \text{ phys } \rangle &= -\sum_a e_a \tilde{p}^a| \text{ phys } \rangle = 0.
\end{aligned}
\tag{2.19}
$$

3. θ-vacuum

When all ratios of various charges e_a are rational, there results a θ-vacuum. In this paper we restrict ourselves to the case in which all fermions have the same charge: $e_a = e$. It is appropriate to introduce the Wilson line phase Θ_W:

$$
\Theta_W = ebL \qquad e^{i\Theta_W(t)} = e^{ie\int_0^L dx\, A_1(t,x)}.
\tag{3.1}
$$

The zero-mode part of the Hamiltonian (2.18) becomes

$$
H_0 = -\frac{\pi N}{6L} + \frac{\pi \mu^2 L}{2N}P_W^2 + \frac{1}{2\pi L}\sum_{a=1}^{N}\{(\Theta_W + 2\pi p_a)^2 + \pi^2 \tilde{p}_a^2\}
\tag{3.2}
$$

where $\mu^2 = Ne^2/\pi$ and $P_W = \dot{\Theta}_W/e^2 L$ is the conjugate momentum to Θ_W. In terms of the new variables

$$
\begin{aligned}
\Theta_W' &= \Theta_W + \frac{2\pi}{N}\sum p_a & q_a' &= q_a + \frac{2\pi}{N}P_W \\
[\Theta_W', P_W] &= i & [q_a', p_b] &= i\delta_{ab} & \text{others} &= 0
\end{aligned}
\tag{3.3}
$$

the Hamiltonian becomes

$$
H_0 = -\frac{\pi N}{6L} + \frac{\pi \mu^2 L}{2N}P_W^2 + \frac{N}{2\pi L}\Theta_W'^2 - \frac{2\pi}{NL}\left(\sum_a p_a\right)^2 + \frac{2\pi}{L}\sum\left(p_a^2 + \frac{1}{4}\tilde{p}_a^2\right).
\tag{3.4}
$$

There appears an additional symmetry $(\Theta_W, p_a) \to (-\Theta_W, -p_a)$ when $\alpha_a = 0$ and $\tfrac{1}{2}$. The Hamiltonian is invariant under

$$
\Theta_W \to \Theta_W + 2\pi \qquad p_a \to p_a - 1
\tag{3.5}
$$

or equivalently, in terms of the original fields,

$$
A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda \qquad \psi_a \to e^{i\Lambda}\psi \qquad \Lambda = \frac{2\pi x}{L}.
\tag{3.6}
$$

The transformation is generated by a unitary operator

$$U = e^{i(2\pi P_W + \sum q_a)} = e^{i \sum q'_a}$$
$$[U, H] = 0. \tag{3.7}$$

In a vector-like theory $\tilde{p}_a = p^a_+ - p^a_-$ takes integer eigenvalues. Further $[\tilde{p}_a, H] = 0$. We can restrict ourselves to states with $\tilde{p}_a = 0$ as the energy is minimized there. The vacuum state is written as a direct product of ground states of the zero-mode sector and oscillatory-mode (ϕ) sector. The ground state in the oscillatory-mode sector is defined with respect to physical boson masses μ_α's. As we shall see, the ground-state wavefunction in the zero-mode sector affects the physical boson masses, and vice versa. These two must be determined self-consistently. Note that if there is a background charge ($Q_{b.g.} \neq 0$) as in the case of spin chains, then $-\sum e_a \tilde{p}_a = -Q_{b.g.}$.

With this understanding the vacuum wavefunction is written as

$$|\Psi_{\text{vac}}\rangle = \int_{-\infty}^{\infty} dp_W \sum_{\{n, r_a\}} |p_W, n, r_a\rangle \tilde{f}(p_W, n, r_a) \tag{3.8}$$

where $|p_W, n, r_a\rangle$ is an eigenstate of P_W and p_a:

$$P_W |p_W, n, r_a\rangle = p_W |p_W, n, r_a\rangle$$

$$p_a |p_W, n, r_a\rangle = \begin{cases} (n + r_a + \alpha_a)|p_W, n, r_a\rangle & \text{for } a < N \\ (n + \alpha_N)|p_W, n, r_a\rangle & \text{for } a = N. \end{cases} \tag{3.9}$$

It follows that

$$e^{ik\theta_W}|p_W, n, r_b\rangle = |p_W + k, n, r_b\rangle$$

$$e^{\pm iq_a}|p_W, n, r_b\rangle = \begin{cases} |p_W, n, r_b \pm \delta_{b,a}\rangle & \text{for } a < N \\ |p_W, n \pm 1, r_b \mp 1\rangle & \text{for } a = N. \end{cases} \tag{3.10}$$

Since U in (3.7) commutes with the Hamiltonian, one can take

$$U|\Psi_{\text{vac}}(\theta)\rangle = e^{+i\theta}|\Psi_{\text{vac}}(\theta)\rangle. \tag{3.11}$$

As $U|p_W, n, r_a\rangle = e^{2\pi i p_W}|p_W, n+1, r_a\rangle$, one finds $\tilde{f}(p_W, n, r_a) = e^{-in\theta + 2\pi i(n + \bar{\alpha})p_W} \tilde{f}(p_W, r_a)$ where $\bar{\alpha} = N^{-1} \sum_{a=1}^{N} \alpha_a$.

This is the θ-vacuum. The existence of the operator U was suspected long ago in [2], though its explicit form was not given. On a circle, U is unambiguously written in terms of P_W and q_a's. This definition was first given in [30]. In passing, it has been noticed recently that a similar definition of the θ-vacuum arises in the framework of light-cone quantization of the Schwinger model [18, 19].

It is convenient to adopt a coherent state basis given by

$$|p_W, n, \varphi_a\rangle = \frac{1}{(2\pi)^{(N-1)/2}} \sum_{\{r_a\}} e^{-i(r_1\varphi_1 + \cdots + r_{N-1}\varphi_{N-1})}|p_W, n, r_a\rangle. \tag{3.12}$$

Then

$$|\Psi_{\text{vac}}(\theta)\rangle = \frac{1}{\sqrt{2\pi}} \sum_n \int dp_W \, [d\varphi]|p_W, n, \varphi_a\rangle e^{-in\theta + 2\pi i(n+\bar{\alpha})p_W} \hat{f}(p_W, \varphi_a)$$

$$\hat{f}(p_W, \varphi_a) = \frac{1}{(2\pi)^{(N-1)/2}} \sum_{\{r_a\}} e^{i \sum r_a \varphi_a} \tilde{f}(p_W, r_a). \tag{3.13}$$

The normalization is $\langle \Psi_{\text{vac}}(\theta')|\Psi_{\text{vac}}(\theta)\rangle = \delta_{2\pi}(\theta' - \theta)$ and $\int dp_W \, [d\varphi]|\hat{f}|^2 = 1$.

The function $\hat{f}(p_W, \varphi_a)$ is determined by solving the eigenvalue equation $H_{\text{tot}}|\Psi_{\text{vac}}(\theta)\rangle = E_{\text{vac}}|\Psi_{\text{vac}}(\theta)\rangle$. We write, for an operator $Q = Q(\Theta_W, P_W, p_a)$,

$$Q|\Psi_{\text{vac}}(\theta)\rangle = \frac{1}{\sqrt{2\pi}} \sum_n \int \mathrm{d}p_W \, [\mathrm{d}\varphi]|p_W, n, \varphi_a\rangle \mathrm{e}^{-\mathrm{i}n\theta + 2\pi\mathrm{i}(n+\bar{\alpha})p_W} \hat{Q}\hat{f}(p_W, \varphi_a). \tag{3.14}$$

Noticing $\Theta_W|p_W, n, \varphi_a\rangle = -\mathrm{i}(\partial/\partial p_W)|p_W, n, \varphi_a\rangle$, one finds that

$$Q = \Theta_W + \frac{2\pi}{N}\sum_{a=1}^{N} p_a \Rightarrow \hat{Q} = \mathrm{i}\left\{\frac{\partial}{\partial p_W} - \frac{2\pi}{N}\sum_{a=1}^{N-1}\frac{\partial}{\partial \varphi_a}\right\}. \tag{3.15}$$

The operator $\hat{H}_0$ corresponding to H_0 is

$$\hat{H}_0 = -\frac{\pi N}{6L} + \frac{e^2 L\, p_W^2}{2} - \frac{N}{2\pi L}\left(\frac{\partial}{\partial p_W} - \frac{2\pi}{N}\sum_{a=1}^{N-1}\frac{\partial}{\partial \varphi_a}\right)^2 - \frac{2\pi(N-1)}{NL}\Delta_\varphi. \tag{3.16}$$

Here the φ-Laplacian is given by

$$\Delta_\varphi = \sum_{a=1}^{N-1}\left(\frac{\partial}{\partial \varphi_a} - \mathrm{i}\beta_a\right)^2 - \frac{2}{N-1}\sum_{a<b}^{N-1}\left(\frac{\partial}{\partial \varphi_a} - \mathrm{i}\beta_a\right)\left(\frac{\partial}{\partial \varphi_b} - \mathrm{i}\beta_b\right) \tag{3.17}$$

where $\beta_a = \alpha_a - \alpha_N$.

The mass operator in the Schrödinger picture is

$$M_{aa}^S = \psi_-^{a\dagger}\psi_+^a = -C_-^{a\dagger}C_+^a \cdot \mathrm{e}^{-2\pi\mathrm{i}\tilde{p}^a x/L}\mathrm{e}^{\mathrm{i}q^a}L^{-1}N_0[\mathrm{e}^{\mathrm{i}\sqrt{4\pi}\phi_a}] \tag{3.18}$$

where the normal ordering $N_0[\ ]$ is defined with respect to massless fields. In the presence of H_{mass}, all boson fields (ϕ) become massive. We denote a mass eigenstate with a mass μ_α by χ_α:

$$\chi_\alpha = U_{\alpha a}\phi_a \qquad U^t U = I. \tag{3.19}$$

The vacuum is defined with respect to these χ_α fields.

With the aid of (B.7), the mass operator becomes

$$M_{aa}^S = -C_-^{a\dagger}C_+^a \cdot \mathrm{e}^{-2\pi\mathrm{i}\tilde{p}^a x/L}\mathrm{e}^{\mathrm{i}q^a}\ L^{-1}\bar{B}_a\prod_{\alpha=1}^{N}N_{\mu_\alpha}[\mathrm{e}^{\mathrm{i}U_{\alpha a}\sqrt{4\pi}\chi_\alpha}]$$

$$\bar{B}_a = \prod_{\alpha=1}^{N}B(\mu_\alpha L)^{(U_{\alpha a})^2}. \tag{3.20}$$

Further

$$\langle p_W', n', \varphi'|\mathrm{e}^{\pm\mathrm{i}q_a}|p_W, n, \varphi\rangle$$

$$= \delta(p_W' - p_W)\prod_{b=1}^{N-1}\delta_{2\pi}(\varphi_b' - \varphi_b)\begin{cases}\delta_{n',n}\mathrm{e}^{\pm\mathrm{i}\varphi_a} & \text{for } a < N \\ \delta_{n',n\pm1}\mathrm{e}^{\mp\mathrm{i}\sum_b \varphi_b} & \text{for } a = N\end{cases} \tag{3.21}$$

so that

$$Q = \mathrm{e}^{\pm\mathrm{i}q_a} \Rightarrow \hat{Q} = \begin{cases}\mathrm{e}^{\pm\mathrm{i}\varphi_a} & \text{for } a < N \\ \mathrm{e}^{\pm\mathrm{i}(\theta - \sum\varphi_b - 2\pi p_W)} & \text{for } a = N.\end{cases} \tag{3.22}$$

Let us write a fermion mass as $m_a = |m_a|\mathrm{e}^{\mathrm{i}\delta_a}$ and drop the absolute value sign henceforth. Then $H_{\text{mass}} = \int \mathrm{d}x \sum_a m_a(M_a\mathrm{e}^{\mathrm{i}\delta_a} + \text{h.c.})$. When H_{mass} acts on $|\Psi_{\text{vac}}(\theta)\rangle$, in general both zero-modes and χ_α fields are excited. In deriving an equation for $\hat{f}(p_W, \varphi)$,

we ignore those χ-excitations, with the understanding that physical masses μ_α's are taken. Then

$$\hat{H}_{\text{mass}} = -\sum_{a=1}^{N-1} 2m_a \bar{B}_a \cos(\varphi_a + \delta_a) - 2m_N \bar{B}_N \cos\left(\theta - \sum \varphi_b - 2\pi p_W + \delta_N\right). \tag{3.23}$$

$\hat{f}(p_W, \varphi_a)$ must satisfy $\hat{H}_{\text{tot}}\hat{f}(p_W, \varphi_a) = E_{\text{vac}}\hat{f}(p_W, \varphi_a)$ where $\hat{H}_{\text{tot}} = \hat{H}_0 + \hat{H}_{\text{mass}}$.

At this stage we recognize that it is appropriate to introduce

$$f(p_W, \varphi_a) = \hat{f}\left(p_W, \varphi_a - \frac{2\pi p_W}{N} - \delta_a\right). \tag{3.24}$$

The eigenvalue equation $\hat{H}_{\text{tot}}\hat{f}(p_W, \varphi_a) = E_{\text{vac}}\hat{f}(p_W, \varphi_a)$ now reads

$$\left\{-\left(\frac{N}{2\pi}\right)^2 \frac{\partial^2}{\partial p_W^2} - (N-1)\Delta_\varphi + V(p_W, \varphi)\right\} f(p_W, \varphi) = \epsilon f(p_W, \varphi) \tag{3.25}$$

where $\epsilon = (NLE_{\text{vac}}/2\pi) + (\pi N^2/12)$. The potential is given by

$$V(p_W, \varphi) = +\frac{(\mu L)^2}{4} p_W^2 - \frac{N}{\pi}\sum_{a=1}^{N} m_a L \bar{B}_a \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right)$$

$$\varphi_N = \theta_{\text{eff}} - \sum_{a=1}^{N-1} \varphi_a \qquad \theta_{\text{eff}} = \theta + \sum_{a=1}^{N} \delta_a. \tag{3.26}$$

The vacuum is

$$|\Psi_{\text{vac}}(\theta)\rangle = \frac{1}{\sqrt{2\pi}}\sum_n \int dp_W \, [d\varphi] |p_W, n, \varphi_a\rangle$$

$$\times e^{-in\theta + 2\pi i(n+\bar{\alpha})p_W} f\left(p_W, \varphi_a + \frac{2\pi p_W}{N} + \delta_a\right). \tag{3.27}$$

The problem has been reduced to solving the Schrödinger equation for N degrees of freedom.

4. Boson masses and condensates

In deriving the equation (3.25) for the vacuum wavefunction $f(p_W, \varphi)$, we have supposed that we already know the masses μ_α's of the boson fields χ_α's. In this section we show how these μ_α's are related to the vacuum wavefunction $f(p_W, \varphi)$ itself. Hence we obtain a self-consistency condition for the vacuum. Along the way we shall also find that the chiral condensate $\langle\bar{\psi}_a\psi_a\rangle$ is related to μ_α's.

From (3.22) it follows that

$$\langle e^{\pm iq_a}\rangle_\theta = \lim_{\theta'\to\theta} \langle\Psi_{\text{vac}}(\theta')|e^{\pm iq_a}|\Psi_{\text{vac}}(\theta)\rangle / \langle\Psi_{\text{vac}}(\theta')|\Psi_{\text{vac}}(\theta)\rangle$$

$$= e^{\mp i\delta_a} \langle e^{\pm i(\varphi_a - 2\pi p_W/N)}\rangle_f \tag{4.1}$$

where φ_N is defined in (3.26) and the f-average is given by $\langle F\rangle_f = \int dp_W \, [d\varphi]F|f|^2$. In our approximation scheme $\langle M_{aa}\rangle_\theta = -(\bar{B}_a/L)\langle e^{iq_a}\rangle_\theta$, and therefore

$$\langle\bar{\psi}_a\psi_a\rangle'_\theta = -\frac{2}{L}\bar{B}_a\left\langle\cos\left(\varphi_a - \frac{2\pi p_W}{N} - \delta_a\right)\right\rangle_f. \tag{4.2}$$

Rigorously speaking, the formula (4.2) is valid only for small m. Adam has determined the condensate to O(m) in mass perturbation theory in the $N = 1$ case [14]. Although the

formula (4.2) incorporates some of the effects of a fermion mass m through $\bar{B}_a$ and $f(\varphi)$, it does not incorporate higher-order radiative corrections which become important for a large $m \gg \mu$. We shall see that our formula gives a fairly good agreement with the lattice result for $m < \mu$ in the $N = 1$ case. Adam's mass perturbation theory fails for $m > 0.5\mu$ [20].

There is ambiguity in the definition of the composite operator $\bar{\psi}\psi(x)$. It diverges in perturbation theory. It has to be normalized such that $\langle\bar{\psi}\psi\rangle = 0$ in a free theory ($e = 0$) in the infinite-volume limit ($L \to \infty$). In other words

$$\langle\bar{\psi}_a\psi_a\rangle_\theta = \langle\bar{\psi}_a\psi_a\rangle_\theta' - \langle\bar{\psi}_a\psi_a\rangle_{\text{free}}'. \tag{4.3}$$

The values of $\langle\bar{\psi}_a\psi_a\rangle_\theta'$ and $\langle\bar{\psi}_a\psi_a\rangle_{\text{free}}'$ depend on the regularization method employed, but the difference does not. In our regularization scheme we shall find $\langle\bar{\psi}_a\psi_a\rangle_{\text{free}}' = -e^{2\gamma}m_a/\pi$. (See (7.3).)

The fermion mass term $H_{\text{mass}} = \int dx \sum_a\{m_a e^{i\delta_a} M_{aa} + (\text{h.c.})\}$ has many effects. In addition to giving a 'potential' in the zero-mode sector as discussed in the previous section, it also gives mass terms ($\propto \chi^2$) and other interactions. It follows from (3.20) that

$$H_{\text{mass}} \Rightarrow -\frac{1}{L}\int dx \sum_a \left\{m_a\bar{B}_a\langle e^{i(\varphi_a-2\pi p_W)/N}\rangle_f \prod_{\alpha=1}^{N} N_{\mu_\alpha}[e^{iU_{\alpha a}\sqrt{4\pi}\chi_\alpha}] + (\text{h.c.})\right\}. \tag{4.4}$$

When fermion masses are small compared with the coupling constant, it is legitimate to expand H_{mass} in power series of χ_α. We define

$$R_a + iI_a = \frac{8\pi}{L}m_a\bar{B}_a \cdot \langle e^{i(\varphi_a-2\pi p_W/N)}\rangle_f. \tag{4.5}$$

It follows that

$$H_{\text{mass}} \Rightarrow \int dx \left\{+\frac{1}{\sqrt{4\pi}}\sum_\alpha\sum_a \chi_\alpha U_{\alpha a} I_a + \frac{1}{2}\sum_{\alpha\beta}\sum_a \chi_\alpha U_{\alpha a} R_a U_{\alpha\beta}^t \chi_\beta\right\}. \tag{4.6}$$

Including the additional mass term coming from the Coulomb interaction, one finds

$$H_{\text{mass}}^\chi = \int dx \left\{+\frac{1}{\sqrt{4\pi}}\chi_\alpha U_{\alpha a} I_a + \frac{1}{2}\chi_\alpha K_{\alpha\beta}\chi_\beta\right\}$$
$$K_{\alpha\beta} = \frac{\mu^2}{N}\sum_{a,b}U_{\alpha a}U_{\beta b} + \sum_a U_{\alpha a}U_{\beta a}R_a. \tag{4.7}$$

$U_{\alpha a}$'s are determined such that $K_{\alpha\beta} = \mu_\alpha^2\delta_{\alpha\beta}$. In other words, we diagonalize

$$\mathcal{K} = \frac{\mu^2}{N}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_N \end{pmatrix}$$
$$K = U\mathcal{K}U^t = \begin{pmatrix} \mu_1^2 & & \\ & \ddots & \\ & & \mu_N^2 \end{pmatrix}. \tag{4.8}$$

The set of equations (3.20), (3.25), (4.5), and (4.8) needs to be solved simultaneously. This is a Hartree–Fock approximation applied to the zero-mode and oscillatory-mode sectors. We call it the generalized Hartree–Fock approximation

In terms of R_a the chiral condensates are, in the case $\delta_a = 0$, $m_a\langle\bar{\psi}_a\psi_a\rangle_\theta' = -R_a/4\pi$. Equations (4.8) and (4.3) relate boson masses to chiral condensates. It is a part of the PCAC (partially conserved axial currents) relations.

As fermion masses become larger, nonlinear terms ($\sim \chi^n$) in H_{mass} become relevant. The boson masses are not simply given by (4.8). Improvement is necessary.
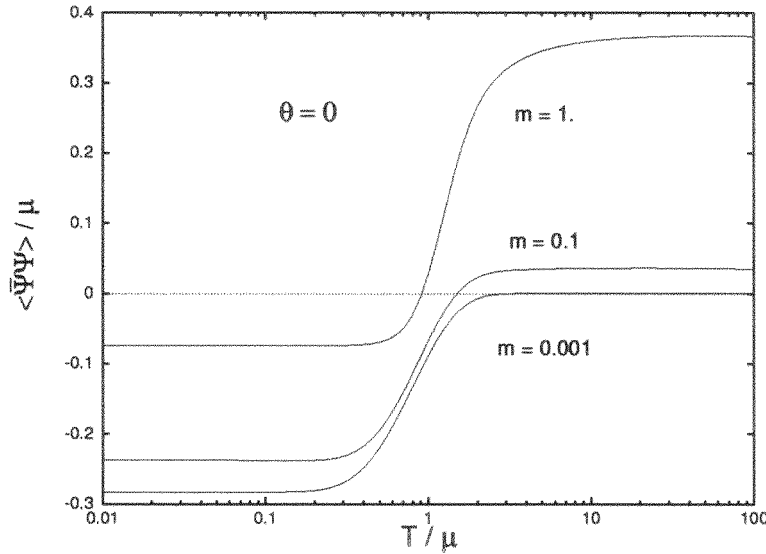
Figure 1. Temperature-dependence of the chiral condensate in the $N = 1$ model at $\theta = 0$. There appears crossover transition around $T/\mu = 1$.

5. Massless fermions ($\alpha_a = \alpha$)

When all fermions are massless and satisfy the same boundary conditions $\alpha_a = \alpha$, the model is exactly solvable. In this case equation (3.25) reduces to

$$
\left\{ -\left(\frac{N}{2\pi}\right)^2 \frac{\partial^2}{\partial p_W^2} + \frac{(\mu L)^2}{4} p_W^2 - (N-1)\Delta_\varphi \right\} f(p_W, \varphi) = \epsilon f(p_W, \varphi)
$$

$$
\Delta_\varphi = \sum_{a=1}^{N-1} \frac{\partial^2}{\partial \varphi_a^2} - \frac{2}{N-1} \sum_{a<b}^{N-1} \frac{\partial^2}{\partial \varphi_a \partial \varphi_b}.
$$

(5.1)

The ground-state, or the vacuum, wavefunction is independent of φ_a. It is given by

$$
f(p_W, \varphi) = \text{constant} \cdot e^{-\pi \mu L p_W^2/2N}.
$$

(5.2)

The boson mass spectrum is given by $\mu_1 \equiv \mu = \sqrt{N}e/\pi$ and $\mu_2 = \cdots = \mu_N = 0$. The orthogonal matrix U in (3.19) has $U_{1a} = 1/\sqrt{N}$ and $\bar{B}_a$ in (3.20) is

$$
\bar{B}_a = \bar{B} = B(\mu L)^{1/N}.
$$

(5.3)

For $N \geqslant 2$, $\langle \cos(\varphi_a - 2\pi p_W/N) \rangle_f = 0$ in (4.2) as $f(p_W, \varphi)$ is independent of φ_a. The chiral condensate $\langle \bar{\psi}_a \psi_a \rangle_\theta$ vanishes for $N \geqslant 2$, reflecting Coleman's theorem which states that in two dimensions a continuous symmetry cannot be spontaneously broken [37]. The nonvanishing $\langle \bar{\psi}_a \psi_a \rangle_\theta$ breaks the $SU(N)$ chiral symmetry. For $N = 1$ the $U(1)$ chiral symmetry is broken by anomaly. Recalling $\varphi_1 = \theta$ in (3.26) for $N = 1$, we have

$$
\langle \bar{\psi}_a \psi_a \rangle_\theta =
\begin{cases}
-\dfrac{2}{L} B(\mu L) e^{-\pi/\mu L} \cos \theta & \text{for } N = 1 \\[2mm]
0 & \text{for } N \geqslant 2.
\end{cases}
$$

(5.4)

The condensate for $N = 1$ is plotted in figure 1.

The $SU(N)$ invariant condensate is nonvanishing, however. It follows from (3.10) that

$$\langle p'_W, n', \varphi' | \exp\left(\pm i \sum_{a=1}^{N} q_a\right) | p_W, n, \varphi \rangle = \delta_{n', n\pm 1} \delta(p'_W - p_W) \prod_{a=1}^{N-1} \delta_{2\pi}(\varphi'_a - \varphi_a). \qquad (5.5)$$

Straightforward calculations in the Schrödinger picture yield

$$\langle \bar{\psi}_{NL} \dots \bar{\psi}_{1L} \psi_{1R} \dots \psi_{NR} \rangle_\theta = \left(\frac{-1}{L}\right)^N \langle e^{-i \sum_a q_a} N_0 [e^{i\sqrt{4\pi N}\chi_1}] \rangle_\theta$$

$$= \left[\frac{-B(\mu L)}{L}\right]^N e^{-i\theta} e^{-N\pi/\mu L}. \qquad (5.6)$$

In the infinite-volume limit it approaches $e^{-i\theta}(-\mu e^\gamma/4\pi)^N$.

6. Truncation

There are two potential terms in the equation for $f(p_W, \varphi)$, equation (3.25). If fermion masses are small compared to μ, one of them, $(\mu L)^2 p_W^2/4$, dominates over the other term in $V(p_W, \varphi)$. The condition is $m_a \bar{B}_a \ll \mu^2 L$. For $m_a \ll \mu$, it is satisfied when $\mu L \gg m/\mu$ $(T/\mu \ll \mu/m)$.

In such a situation P_W acts as a fast variable, whereas the φ_a's act as slow variables. The wavefunction can be approximated by $f(p_W, \varphi) = \sum_{s=0}^{\infty} u_s(p_W) f_s(\varphi) \sim u(p_W) f(\varphi)$ where $u(p_W) = u_0(p_W)$ is

$$\left\{-\left(\frac{N}{2\pi}\right)^2 \frac{d^2}{dp_W^2} + \frac{(\mu L)^2}{4} p_W^2\right\} u(p_W) = \frac{N\mu L}{4\pi} u(p_W)$$

$$u(p_W) = \left(\frac{\mu L}{N}\right)^{1/4} e^{-\pi\mu L p_W^2/2N}. \qquad (6.1)$$

The cosine term in $V_N(p_W, \varphi)$ is approximated by

$$\cos\left(\varphi_a - \frac{2\pi p_W}{N}\right) \to \int dp_W \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right) u(p_W)^2 = e^{-\pi/N\mu L} \cos\varphi_a. \qquad (6.2)$$

We remark that the truncated equation has symmetry $p_W \to -p_W$, although the original equation does not in general. The truncation may not be good for physical quantities sensitive to this symmetry.

The function $f(\varphi)$ satisfies

$$\{-(N-1)\Delta_\varphi + V_N(\varphi)\} f(\varphi) = \epsilon_0 f(\varphi)$$

$$V_N(\varphi) = -\frac{NL}{\pi} e^{-\pi/N\mu L} \sum_{a=1}^{N} m_a \bar{B}_a \cos\varphi_a. \qquad (6.3)$$

This equation has been extensively studied in the $N = 2$ and $N = 3$ cases in [10, 11].

Let us denote $\langle\langle F(\varphi)\rangle\rangle_f = \int [d\varphi] F(\varphi) |f(\varphi)|^2$. Then (4.1) and (4.5) become

$$\langle e^{\pm i q_a}\rangle_\theta = e^{\mp i\delta_a} e^{-\pi/N\mu L} \langle\langle e^{\pm i\varphi_a}\rangle\rangle_f$$

$$R_a + iI_a = \frac{8\pi}{L} m_a \bar{B}_a e^{-\pi/N\mu L} \langle\langle e^{i\varphi_a}\rangle\rangle_f. \qquad (6.4)$$

All formulae in section 4 remain intact with these substitutions. In particular, the chiral condensate satisfies $\langle \bar{\psi}_a \psi_a \rangle'_\theta = -R_a/4\pi m_a$.

7. $N = 1$ massive case

7.1. Generalized Hartree–Fock approximation

With one fermion there is no φ degree of freedom. We write $m_1 = m, \delta_1 = 0, \alpha_1 = \alpha$. Recall that $\mu = e/\sqrt{\pi}$ and $\varphi_1 = \theta_{\text{eff}}$. The vacuum wavefunction is determined by

$$|\Psi_{\text{vac}}(\theta)\rangle = \frac{1}{\sqrt{2\pi}} \sum_n \int dp_W \, |p_W, n\rangle e^{-in\theta + 2\pi i(n+\alpha)p_W} f(p_W)$$

$$\left\{ -\frac{1}{(2\pi)^2} \frac{d^2}{dp_W^2} + \frac{(\mu L)^2}{4} p_W^2 - \frac{mLB(\mu_1 L)}{\pi} \cos(2\pi p_W - \theta_{\text{eff}}) \right\} f(p_W) = \epsilon f(p_W). \tag{7.1}$$

When $m \ll \mu$, the boson mass μ_1 must satisfy

$$\mu_1^2 = \mu^2 + \frac{8\pi m B(\mu_1 L)}{L} \langle \cos(2\pi p_W - \theta)\rangle_f. \tag{7.2}$$

As m becomes larger, the formula (7.2) needs to be improved to incorporate nonlinear effects in the fermion mass m. In particular, $\mu_1 = 2m + \mathrm{O}(e^2/m)$ for $m \gg \mu$, as the boson is interpreted as a fermion–antifermion bound state.

In determining a physical chiral condensate, one needs to subtract a condensate in free theory as discussed in section 4. The 'free' limit corresponds to the limit $m \gg \mu$. At the moment we do not have reliable formulae which relate μ_1, m, and $\langle \bar{\psi}\psi \rangle'_\theta$ for $m \gg \mu$. The best we can do is to extrapolate (4.2) and (7.2) for large m to determine the subtraction term within our approximation.

It follows from (7.1) that in the weak coupling $e/m \ll 1$ and infinite volume $L \to \infty$ limits $\langle \cos(2\pi p_W - \theta)\rangle_f = 1$. If the formula (7.2) is employed, one obtains $\mu_1 = 2e^\gamma m$. Combined with (4.2), it gives

$$\langle \bar{\psi}\psi \rangle'_{\text{free}} = -\frac{e^{2\gamma}}{\pi} m. \tag{7.3}$$

The chiral condensate is therefore given by

$$\langle \bar{\psi}\psi \rangle_\theta = -\frac{2B(\mu_1 L)}{L} \langle \cos(2\pi p_W - \theta)\rangle_f + \frac{e^{2\gamma}}{\pi} m \tag{7.4}$$

which we expect to be a good approximation for $m < \mu$. We stress that within our approximation the subtraction term is given by (7.3). In an exact treatment the boson mass in the weak coupling limit should be given by $\mu_1 = 2m$. To achieve this, one has to improve both (4.2) and (7.2) consistently.

In the massless case ($m = 0$), $\langle \bar{\psi}\psi \rangle_\theta^{m=0} = -2L^{-1} B(\mu L) e^{-\pi/\mu L} \cos\theta$. In the infinite-volume limit or zero-temperature limit, it approaches $-(\mu e^\gamma/2\pi) \cos\theta$.

It may be of interest to apply a perturbation theory [12] to (7.1) when $m \ll \mu$. Write (7.1) in the form

$$(H_0 + V)|\Psi\rangle = E|\Psi\rangle$$
$$H_0 = 2\pi\mu L(a^\dagger a + \tfrac{1}{2}) \tag{7.5}$$
$$V = -\kappa \cos(2\pi p_W - \theta) \qquad \kappa = 4\pi m L B(\mu_1 L)$$

where the annihilation and creation operators are

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\pm \frac{1}{\sqrt{\pi\mu L}} \frac{d}{dp_W} + \sqrt{\pi\mu L} \, p_W \right). \tag{7.6}$$

In terms of number eigenstates $|n\rangle$, $|\Psi\rangle$ is found to be, to $O(m/\mu)$,

$$|\Psi\rangle = |0\rangle - \frac{1}{2\pi\mu L}\sum_{n=1}^{\infty}\frac{1}{n}|n\rangle\langle n|V|0\rangle. \tag{7.7}$$

Since $\cos(2\pi p_W - \theta) = \frac{1}{2}e^{-\pi/\mu L}(e^{-i\theta}e^{i\alpha a^\dagger}e^{i\alpha a} + \text{h.c.})$ where $\alpha = (2\pi/\mu L)^{1/2}$, it follows that $\langle\cos(2\pi p_W - \theta)\rangle_f^{(0)} = e^{-\pi/\mu L}\cos\theta$ and

$$\begin{aligned}
\langle\cos(2\pi p_W - \theta)\rangle_f^{(1)} &= \frac{\kappa}{\pi\mu L}\sum_{n=1}^{\infty}\frac{1}{n}|\langle 0|\cos(2\pi p_W - \theta)|n\rangle|^2 \\
&= \frac{\kappa}{2\pi\mu L}e^{-2\pi/\mu L}\sum_{n=1}^{\infty}\frac{1 + (-1)^n\cos 2\theta}{n\cdot n!}\left(\frac{2\pi}{\mu L}\right)^n \\
&= \frac{\kappa}{2\pi\mu L}e^{-2\pi/\mu L}\left\{\int_0^{2\pi/\mu L}dz\,\frac{e^z - 1}{z} + \cos 2\theta\int_0^{2\pi/\mu L}dz\,\frac{e^{-z} - 1}{z}\right\} \\
&= e^\gamma\frac{m}{\mu}(1 - \cos 2\theta) \qquad \text{for } \mu L \gg 1.
\end{aligned} \tag{7.8}$$

The boson mass $\mu_1 = \mu + \delta\mu$ is obtained from (7.2):

$$\begin{aligned}
\delta\mu &= \frac{4\pi}{\mu L}mB(\mu L)e^{-\pi/\mu L}\cos\theta + O(m^2/\mu^2) \\
&\sim me^\gamma\cos\theta \qquad \text{for } \mu L \gg 1.
\end{aligned} \tag{7.9}$$

The chiral condensate is given by

$$\begin{aligned}
\langle\bar\psi\psi\rangle_\theta' &= -\frac{2}{L}\{B(\mu L) + B'(\mu L)\delta\mu L\}e^{-\pi/\mu L}\cos\theta - \frac{2}{L}B(\mu L)\langle\cos(2\pi p_W - \theta)\rangle_f^{(1)} \\
&= -\frac{e^\gamma}{2\pi}\mu\cos\theta + \frac{e^{2\gamma}}{4\pi}(-3 + \cos 2\theta)m \qquad \text{for } \mu L \gg 1.
\end{aligned} \tag{7.10}$$

With (7.3)

$$\langle\bar\psi\psi\rangle_\theta \sim -\frac{e^\gamma}{2\pi}\mu\cos\theta + \frac{e^{2\gamma}}{4\pi}(1 + \cos 2\theta)m. \tag{7.11}$$

This result differs from the result in the mass perturbation theory for the reason explained above and should not be taken seriously. Our formalism, however, allows us to estimate $\langle\bar\psi\psi\rangle_{\text{free}}$, and therefore the physical condensate $\langle\bar\psi\psi\rangle_\theta$. The mass perturbation theory is valid only for small $m/\mu \ll 1$, whereas our formalism allows the numerical evaluation of various physical quantities even for $m/\mu \sim 1$. We shall see below a good agreement between ours and the lattice gauge theory in the range $m/\mu < 1$. See section (7.3) below.

7.2. Mass perturbation theory

Corrections can be evaluated in a power series in m/e. This analysis was carried out at zero temperature by Adam [14]. We present the analysis at finite temperature.

We illustrate the computation for the chiral condensate. Recall $\bar\psi\psi = -B(\mu L)L^{-1}(e^{iq}K_+ + e^{-iq}K_-)$ where $K_\pm = e^{\pm i\sqrt{4\pi}\phi^{(-)}}e^{\pm i\sqrt{4\pi}\phi^{(+)}}$. We have suppressed irrelevant factors. In the invariant perturbation theory

$$\langle\bar\psi\psi\rangle_c^{(1)} = -im\left\{\frac{B(\mu L)}{L}\right\}^2\int d^2x\sum_{a,b=\pm}\langle T[e^{iaq(t)}e^{ibq(0)}]T[K_a(x)K_b(0)]\rangle_c. \tag{7.12}$$

Making use of (C.5) and the identity $\langle e^{i\alpha\phi^{(+)}(x)}e^{i\beta\phi^{(-)}(y)}\rangle = e^{-\alpha\beta[\phi^{(+)}(x),\phi^{(-)}(y)]}$, one finds

$$\langle\bar{\psi}\psi\rangle_c^{(1)} = -\frac{2im}{L^2}B(\mu L)^2 e^{-2\pi/\mu L}\int d^2x\ \{(e^{4\pi iG(x)} - 1) + \cos 2\theta(e^{-4\pi iG(x)} - 1)\}$$

$$G(x) = \frac{1}{2\pi L}\sum_n\int_{-\infty}^{\infty}d\omega\ \frac{e^{-i\omega t + ip_n x}}{\omega^2 - p_n^2 - \mu^2 + i\epsilon}.$$

(7.13)

The disconnected component has been subtracted. Deforming the t-integral to the imaginary axis, one finds

$$\langle\bar{\psi}\psi\rangle_c^{(1)} = -\frac{m}{\pi^2}B(\mu L)^2 e^{-(2\pi/\mu L)}$$

$$\times \int_0^{\infty}d\tau\int_0^{2\pi}dx\ \{(e^{+E(\tau,x;\mu L/2\pi)} - 1) + \cos 2\theta(e^{-E(\tau,x;\mu L/2\pi)} - 1)\}$$

$$\equiv [F_+(\mu L) + F_-(\mu L)\cos 2\theta]m$$

(7.14)

$$E(\tau, x; z) = \frac{1}{z}e^{-z\tau} + 2\sum_{n=1}^{\infty}\frac{1}{v_n}e^{-v_n\tau}\cos nx \qquad v_n = (n^2 + z^2)^{1/2}.$$

The coefficient $F_-(\mu L)$ is finite, whereas $F_+(\mu L)$ diverges logarithmically near $\tau = x = 0$. The divergence is due to the $O(m)$ correction in the free ($e = 0$) theory, which must be subtracted to define the physical chiral condensate as explained in section 4. Hence in the mass perturbation theory

$$\langle\bar{\psi}\psi\rangle_\theta = -\frac{e^\gamma}{2\pi}\mu\cos\theta + m[F_+(\mu L) - F_+^{\text{free}} + F_-(\mu L)\cos 2\theta] + \cdots.$$

(7.15)

F_+^{free} is obtained in a similar manner. One starts with a massless free fermion theory. In the bosonization method the vacuum satisfies $p|\text{vac}\rangle = 0$. Since $\langle e^{\pm iq}\rangle = 0$, the condensate vanishes; $\langle\bar{\psi}\psi\rangle = 0$. The fermion mass is treated as a perturbation. To $O(m)$ one gets an expression which is the same as (7.12) except that $B(\mu L)$ is replaced by 1 and $\phi(x)$ represents a massless field. Making use of (C.2), one finds that

$$\langle\bar{\psi}\psi\rangle_{\text{free}}^{(1)} = -\frac{2im}{L^2}\int d^2x\ e^{-2\pi i|t|/L}e^{4\pi iG_{\text{free}}(x)}$$

(7.16)

where $G_{\text{free}}(x) = -i\langle T[\phi(x)\phi(0)]\rangle$ is a massless propagator excluding the contribution from the zero mode. Employing (B.1) and deforming the integration path, one finds

$$\langle\bar{\psi}\psi\rangle_{\text{free}}^{(1)} = -\frac{m}{\pi^2}\int_0^{\infty}d\tau\int_0^{2\pi}dx\ \frac{e^{-\tau}}{(1 - e^{-\tau-ix})(1 - e^{-\tau+ix})} \equiv mF_+^{\text{free}}.$$

(7.17)

Notice that F_+^{free} is independent of L. Comparing (7.17) with (7.14), we observe that $F_+^{\text{free}} = F_+(\mu L)|_{\mu L\to 0}$.

$F_-(\mu L)$ can be easily evaluated by numerical integration. At $\mu L/2\pi = 10 \sim 50$, $F_- = 0.357$. This is consistent with the number, 0.3581, obtained by Adam in the $L \to \infty$ limit. The divergence in $F_+(\mu L)$ and F_+^{free} makes the evaluation of the difference very difficult. We comment that Adam's subtraction procedure to get finite F_+^{free} is inconsistent. Indeed, his massless propagator differs from that obtained by taking the $\mu \to 0$ limit of the massive propagator in [14]. Further, his numerical estimate, $-0.391\,26$, for $F_+(\infty) - F_+^{\text{free}}$ disagrees with the lattice result. At $\theta = 0$ Adam's estimate gives $F_+(\infty) - F_+^{\text{free}} + F_-(\infty) = -0.0345 < 0$ in equation (7.15), which contradicts with the recent result from the lattice gauge theory. (See figure 2.) There is a disagreement in the sign.
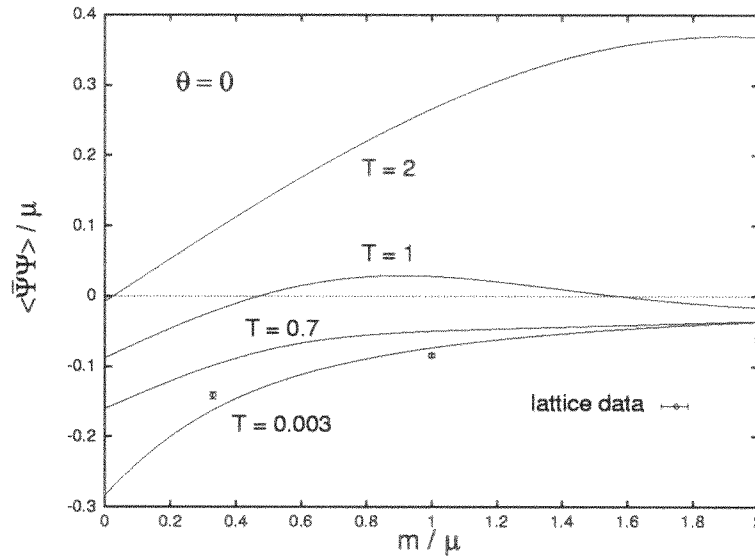
Figure 2. The mass-dependence of the chiral condensate in the $N = 1$ model at $\theta = 0$ with various values of temperature T. In the figure T is in a unit of μ. The lattice data, which corresponds to $T = 0$, is by de Forcrand *et al* [23]. The additional lattice data point at $m/\mu = 2$ was provided by the authors of [23]. The condensate for $T/\mu < 0.1$ is essentially the same as that at $T = 0$. The curve for $T/\mu = 0.003$ is consistent with the result by Tomachi and Fujita [16].

7.3. Numerical evaluation in the generalized Hartree–Fock approximation

In this section we present various results obtained by numerical evaluation. The algorithm is simple. With given μL, m/μ, and θ_{eff}, we start with an initial μ_1/μ. Then equation (7.1) is solved numerically to find $f(p_W)$. With this $f(p_W)$, a new μ_1/μ is determined by (7.2). We have a mapping

$$\mu_1 \to f(p_W) \to \mu_1. \tag{7.18}$$

We repeat this process until the output μ_1 coincides with the input μ_1 within required accuracy. With μ_1 being fixed, the condensate is evaluated by (7.4).

In figure 2 the m-dependence of the condensate is plotted at $\theta = 0$ with various values of T/μ. The lattice result from [23] is also plotted for comparison. Our result agrees well with Tomachi and Fujita's evaluation by the Bogoliubov transformation [16]. The agreement with the lattice result is modest. Note that the subtraction of condensates in free theory in each regularization scheme is crucial.

There appears a singularity in the m-dependence of the condensate when θ is close to π. In figure 3 the condensates are plotted with various values of θ. The discontinuity appears at $m/\mu = 0.44$ for $\theta = \pi$, and at $m/\mu = 0.40$ for $\theta = 0.95\pi$.

The discontinuity persists as long as the temperature is lower than $T_c \sim 0.12\mu$. The condensate at various values of T is depicted in figure 4.

The origin of the discontinuity is understood as follows [12]. For $\mu L \gg 1$ equation (7.1) becomes

$$\left\{ -\frac{1}{(2\pi)^2} \frac{\mathrm{d}^2}{\mathrm{d}p_W^2} + \frac{(\mu L)^2}{4} p_W^2 - \frac{m\mu_1 L^2 \mathrm{e}^\gamma}{4\pi^2} \cos(2\pi p_W - \theta_{\text{eff}}) \right\} f(p_W) = \epsilon f(p_W). \tag{7.19}$$
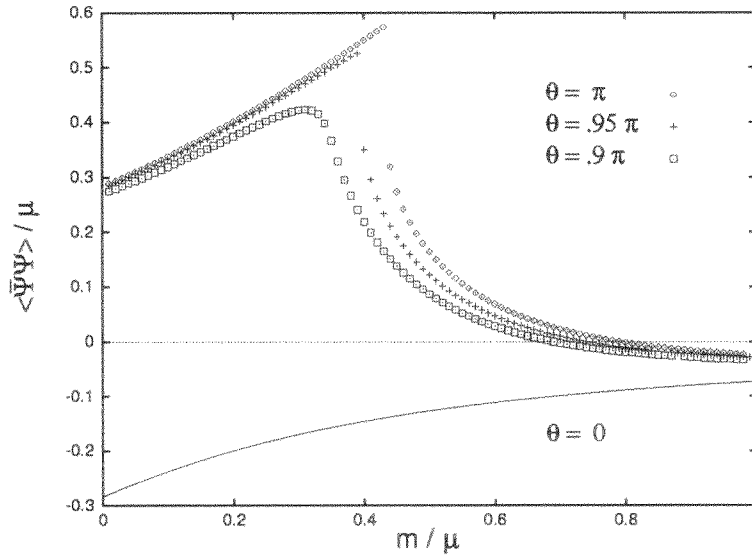
Figure 3. The mass-dependence of the chiral condensate in the $N = 1$ model near $\theta = \pi$ at $T/\mu = 0.003$. There appears a discontinuity above $\theta \sim 0.95\pi$. The condensate at $\theta = 0$ is also displayed for comparison.
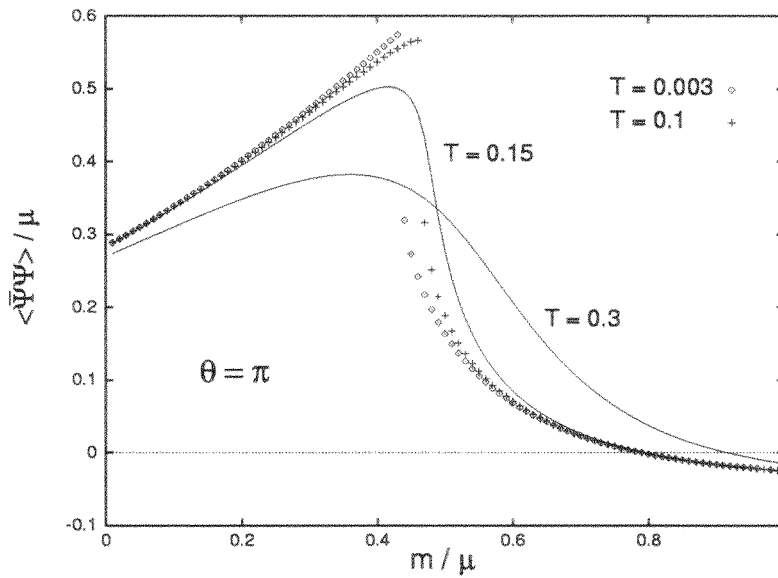

Figure 4. The mass-dependence of the chiral condensate in the $N = 1$ model at $\theta = \pi$ at various values of T/μ. For $T/\mu < 0.12$ the discontinuity remains, whereas the transition becomes smooth for $T/\mu > 0.12$.

The potential term dominates over the kinetic energy term. Suppose that $\theta_{\mathrm{eff}} = \pi$. Then the wavefunction is sharply localized around the absolute minimum of $\hat{V}(p_W) = (\pi\mu p_W)^2 + m\mu_1 e^\gamma \cos(2\pi p_W)$. There is always a solution for which $\mu^2 > 2e^\gamma m\mu_1$. In this case $\hat{V}(p_W)$ is minimized at $p_W = 0$ and $\mu_1 = \sqrt{\mu^2 + m^2 e^{2\gamma}} - m e^\gamma$. There is another

solution for $m/\mu > 0.435$, or $\mu^2 < 2e^\gamma m\mu_1$. $\hat{V}(p_W)$ is minimized at $p_W = \bar{p}_W$ where $2\pi \bar{p}_W = (2m\mu_1 e^\gamma/\mu^2)\sin 2\pi \bar{p}_W$. The boson mass is given by $\mu_1 = \sqrt{\mu^2 + a^2} - a$ where $a = me^\gamma \cos 2\pi \bar{p}_W$. The second solution has a lower energy density and corresponds to the vacuum. Hence at $T = L^{-1} = 0$, there appears a discontinuity at $m_c/\mu = 0.435$.

At finite temperature the critical value m_c/μ is determined numerically. $m_c/\mu = 0.437$ at $T/\mu = 0.03$. The discontinuity disappears for $T/\mu > 0.12$. As noted in [12] there are two possible scenarios. In the full theory the discontinuity may persist, but with a universal m_c/μ independent of T/μ. Or the discontinuity may be smoothed by higher-order corrections. At the moment we do not know for sure which picture is correct.

8. Degenerate fermions

When all fermions have degenerate masses ($m_a = m$) and obey the same boundary conditions $\alpha_a = \alpha$, the Laplacian Δ_φ and potential $V_N(p_W, \varphi)$ in the eigenvalue equation (3.25) become

$$\Delta_\varphi = \sum_{a=1}^{N-1} \frac{\partial^2}{\partial\varphi_a^2} - \frac{2}{N-1} \sum_{a<b}^{N-1} \frac{\partial^2}{\partial\varphi_a \partial\varphi_b}$$

$$V(p_W, \varphi) = +\frac{(\mu L)^2}{4} p_W^2 - \frac{NmL\bar{B}}{\pi} \sum_{a=1}^{N} \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right) \tag{8.1}$$

$$\bar{B} = B(\mu_1 L)^{1/N} B(\mu_2 L)^{1-(1/N)}.$$

Boson masses μ_1 and $\mu_2 = \cdots = \mu_N$ are determined by (4.5)–(4.8);

$$\mu_1^2 = \mu^2 + R \qquad \mu_2^2 = R$$

$$R = \frac{8\pi m\bar{B}}{L} \left\langle \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right)\right\rangle_f. \tag{8.2}$$

When $\mu L, \mu_1 L, \mu_2 L \gg 1$, the potential is approximated by

$$V_\infty = \frac{L^2}{4\pi^2} \left\{ (\pi\mu p_W)^2 - Ne^\gamma m\mu_1^{1/N}\mu_2^{(N-1)/N} \sum_{a=1}^{N} \cos\left(\varphi_a - \frac{2\pi p_W}{N}\right)\right\}. \tag{8.3}$$

If fermion masses are small $m/\mu \ll 1$, the first term in the potential dominates over the second. The p_W-dependence of the wavefunction is the same as in the massless case, and one can write $f(p_W, \varphi) = e^{-\pi\mu L P_W^2/2N} f(\varphi)$. With the aid of the truncation formula (6.2) the equation is reduced to

$$\{-\Delta_\varphi + \kappa F_N(\varphi)\} f(\varphi) = \epsilon f(\varphi)$$

$$\kappa = \frac{N}{\pi(N-1)} mL\bar{B}e^{-\pi/N\mu L} \qquad F_N(\varphi) = -\sum_{a=1}^{N} \cos\varphi_a. \tag{8.4}$$

The boson mass μ_2 is given by

$$\mu_2^2 = \frac{8\pi m\bar{B}}{L} e^{-\pi/N\mu L}\langle\langle\cos\varphi_a\rangle\rangle_f = \frac{8\pi^2(N-1)}{NL^2}\kappa\langle\langle\cos\varphi_a\rangle\rangle_f. \tag{8.5}$$

In the $L \to \infty$ or $T \to 0$ limit, the wavefunction has a sharp peak at the location of the minimum of $V_\infty(p_W, \varphi)$ in (8.3) or $F_N(\varphi)$ in (8.4). We examine the location of the minimum of the potential.

8.1. Potential

(a) $N = 2$

In the two-flavour case $\varphi_1 = \varphi$, $\varphi_2 = \theta - \varphi$ and the potential takes the form of

$$V_\infty(p_W, \varphi) = \frac{L^2}{4\pi^2} \left\{ (\pi\mu p_W)^2 - 4\mathrm{e}^\gamma m \mu_1^{1/2} \mu_2^{1/2} \cos\left(\pi p_W - \frac{\theta}{2}\right) \cos\left(\varphi - \frac{\theta}{2}\right) \right\} \tag{8.6}$$

or

$$F_2(\varphi) = -2 \cos\frac{\theta}{2} \cos\left(\varphi - \frac{\theta}{2}\right). \tag{8.7}$$

The form of the potential V_∞ suggests that the anomalous behaviour analogous to that in the $N = 1$ case may develop near $\theta = \pi$ with $m = \mathrm{O}(\mu)$. Note that at $\theta = \pm\pi$ the truncated equation (8.4) is not valid as the potential $F_2(\varphi) = 0$. The p_W degree of freedom must be retained.

$F_2(\varphi)$ has a minimum at

$$\varphi = \frac{\bar\theta}{2} \qquad \bar\theta = \theta - 2\pi \left[\frac{\theta + \pi}{2\pi}\right] \tag{8.8}$$

where $[x]$ denotes a maximum integer not exceeding x so that $|\bar\theta| \leqslant \pi$. Notice that the location of the minimum discontinuously changes at $\theta = \pi \pmod{2\pi}$.

(b) $N = 3$

Let us examine the potential $F_3(\varphi)$.

$$F_3(\varphi) = -\cos(\varphi_1 + \varphi_2 - \theta) - \cos\varphi_1 - \cos\varphi_2. \tag{8.9}$$

The location of the minimum can be easily found analytically. There are six distinct stationary points: $\varphi_1 = \varphi_2 = \frac{1}{3}\theta, \frac{1}{3}(\theta \pm 2\pi)$, or $(\varphi_1, \varphi_2) = (\theta + \pi, \theta + \pi)$, $(\theta + \pi, -\theta)$, $(-\theta, \theta + \pi)$. The global minimum is located at $\varphi_1 = \varphi_2 = \frac{1}{3}\bar\theta$. The periodicity in θ is 2π. The location of the minimum jumps from $(\varphi_1, \varphi_2) = (+\frac{1}{3}\pi, +\frac{1}{3}\pi)$ to $(-\frac{1}{3}\pi, -\frac{1}{3}\pi)$ at $\theta = \pi \pmod{2\pi}$. The minimum is always located at $|\varphi_1| = |\varphi_2| \leqslant \frac{1}{3}\pi$.

(c) General N

First notice that

$$F_N(\varphi_1, \ldots, \varphi_{N-1}; \theta) = -\sum_{a=1}^{N-1} \cos\varphi_a - \cos\left(\theta - \sum_{a=1}^{N-1} \varphi_a\right)$$

$$= F_{N-1}(\varphi_1, \ldots, \varphi_{N-2}; \theta - \varphi_{N-1}) - \cos\varphi_{N-1}. \tag{8.10}$$

For $N = 3$ we know the minimum is located at $\varphi_1 = \varphi_2 = \frac{1}{3}\bar\theta$.

Consider $N = 4$. We denote the minimum of F_4 by (a_1, a_2, a_3). Fix the value of φ_3 and consider $F_4(\varphi)_{\varphi_3\text{fixed}} \equiv G_4(\varphi_1, \varphi_2)$. Denote the minimum of G_4 by (b_1, b_2) where $b_j = b_j(\varphi_3)$. It follows from (8.10) that (b_1, b_2) is the minimum of $F_3(\varphi_1, \varphi_2; \theta - \varphi_3)$. The result in the $N = 3$ case implies that $b_1 = b_2$. Since the minimum of $H_4(\varphi_3) = G_4(b_1[\varphi_3], b_2[\varphi_3]; \varphi_3)$, which we denote by c, is the minimum of $F_4(\varphi)$, we have $[a_1, a_2, a_3] = [b_1(c), b_2(c), c]$. In particular, $a_1 = a_2$ as $b_1 = b_2$. We repeat the argument with the value of φ_1 kept fixed, to obtain $a_2 = a_3$. Hence $a_1 = a_2 = a_3$, i.e. the minimum of $F_4(\varphi)$ occurs at $\varphi_a = \varphi$.

By induction we conclude that $F_N(\varphi)$ has the minimum at $\varphi_a = \varphi$ ($a = 1, \ldots, N - 1$). It is easy to find the location of the minimum of $\tilde{F}_N(\varphi) = F_N(\varphi, \ldots, \varphi; \theta)$. From the symmetry $\varphi_N = \theta - (N-1)\varphi = \varphi \pmod{2\pi}$, or $\varphi = \theta/N \pmod{2\pi/N}$. Direct evaluation of $\tilde{F}_N(\varphi)$ shows that the minimum of $F_N(\varphi)$ is attained at $\varphi_a = \bar\theta/N$.

8.2. Boson masses and condensates

In a few limiting cases boson masses and chiral condensates can be determined analytically. In this section we suppose that fermion masses are small $m \ll \mu$ and analyse (8.4). The wavefunction $f(\varphi)$ is determined by two parameters, θ and κ. The chiral condensates are related to the boson mass by

$$
\langle \bar{\psi}_a \psi_a \rangle_\theta = \langle \bar{\psi}_a \psi_a \rangle'_\theta - \langle \bar{\psi}_a \psi_a \rangle'_{\text{free}}
$$
$$
\langle \bar{\psi}_a \psi_a \rangle'_\theta = -\frac{\mu_2^2}{4\pi m}.
$$
(8.11)

(a) $N = 2$

We suppose that $\theta \neq \pi$. Equation (8.4) becomes

$$
\left\{ -\frac{d^2}{d\varphi^2} - \kappa_0 \cos\left(\varphi - \frac{\bar{\theta}}{2}\right) \right\} f(\varphi) = \epsilon f(\varphi)
$$
$$
\kappa_0 = 2\kappa \cos \frac{1}{2}\bar{\theta} = \frac{4mL}{\pi} B(\mu_1 L)^{1/2} B(\mu_2 L)^{1/2} e^{-\pi/2\mu L} \cos \frac{\bar{\theta}}{2}.
$$
(8.12)

It is easy to see

$$
\langle\langle \cos \varphi \rangle\rangle_f =
\begin{cases}
\kappa_0 \cos \frac{1}{2}\bar{\theta} = \kappa(1 + \cos \bar{\theta}) & \text{for } \kappa_0 \ll 1 \\
\cos \frac{1}{2}\bar{\theta} & \text{for } \kappa_0 \gg 1
\end{cases}
$$
(8.13)

and accordingly $\mu_2 L / \sqrt{2}\pi = \kappa_0$ or $\sqrt{\kappa_0}$ for $\kappa_0 \ll 1$ or $\kappa_0 \gg 1$, respectively. Hence one finds for $m \ll \mu$

$$
\mu_2 =
\begin{cases}
4\sqrt{2}m \cos \frac{1}{2}\bar{\theta} e^{-\pi/2\mu L} & \text{for } \mu L \ll 1 \\
4\sqrt{2}m \cos \frac{1}{2}\bar{\theta} \left(\frac{e^\gamma \mu L}{4\pi}\right)^{1/2} & \text{for } \mu L \gg 1 \gg (m \cos \frac{1}{2}\bar{\theta})^{2/3} \mu^{1/3} L \\
(2e^\gamma m \mu^{1/2} \cos \frac{1}{2}\bar{\theta})^{2/3} & \text{for } (m \cos \frac{1}{2}\bar{\theta})^{2/3} \mu^{1/3} L \gg 1.
\end{cases}
$$
(8.14)

Coleman obtained $\mu_2 \propto m^{2/3} \mu^{1/3} \cos^{2/3} \frac{1}{2}\bar{\theta}$ in Minkowski spacetime long ago, [4] but the overall coefficient was not determined. Note that if the massless limit is taken with a fixed L, then $\mu_2 = O(m)$.

At $\theta = 0$ and $L \to \infty$, m-dependence of μ_2 has been determined in the lattice gauge theory [22]. In this limit the formula (8.14) leads to $\mu_2/e = 2^{5/6} e_E^{2\gamma/3} \pi^{-1/6} (m/e)^{2/3} = 2.163(m/e)^{2/3}$ where e and e_E are the coupling constant and Euler's constant, respectively. Smilga showed that the exact coefficient should be 2.008 for $m/e \ll 1$ [38]. The lattice simulation is done for $m/e < 0.5$. The data supports the $m^{2/3}$-dependence and indicates a coefficient between the two numbers mentioned above.

(b) $N \geqslant 3$ flavour

When $\kappa \gg 1$ in equation (8.4), the wavefunction has a sharp peak at the global minimum $\varphi_a = \bar{\theta}/N$, and therefore $\langle\langle e^{i\varphi_a} \rangle\rangle_f = e^{i\bar{\theta}/N}$.

For $\kappa \ll 1$ we solve (8.4) in a power series in κ. To O(1) a plane-wave solution $u(\varphi; \vec{n}) = e^{in_1\varphi_1 + \cdots + in_{N-1}\varphi_{N-1}}$ satisfies $-\Delta_N^\varphi u = \epsilon u$ where ϵ is positive semidefinite and vanishes only if $n_1 = \cdots = n_{N-1} = 0$. Hence to O($\kappa^0$) $f^{(0)} = 1$. To find O(κ) correction, we note $\Delta_N^\varphi F_N(\varphi) = -F_N(\varphi)$, from which it follows that $f = 1 - \kappa F_N(\varphi)$. Hence

$$
\langle\langle \cos \varphi_a \rangle\rangle_f =
\begin{cases}
\kappa & \text{for } \kappa \ll 1 \\
\cos \frac{\bar{\theta}}{N} & \text{for } \kappa \gg 1.
\end{cases}
$$
(8.15)

Notice that it is independent of θ for $\kappa \ll 1$. It follows that $[N/8\pi^2(N-1)]^{1/2}\mu_2 L = \kappa$ for $\kappa \ll 1$ and $= [\kappa \cos(\bar\theta/N)]^{1/2}$ for $\kappa \gg 1$. The boson mass is determined from (8.5);

$$
\mu_2 = \begin{cases}
\left(\dfrac{8N}{N-1}\right)^{1/2} m \mathrm{e}^{-\pi/N\mu L} & \text{for } \mu L \ll 1 \\[2ex]
\left(\dfrac{8N}{N-1}\right)^{1/2} m \left(\dfrac{\mathrm{e}^\gamma \mu L}{4\pi}\right)^{1/N} & \text{for } \mu L \gg 1 \gg m^{N/(N+1)}\mu^{1/(N+1)}L \\[2ex]
\left(2\mathrm{e}^\gamma m \mu^{1/N} \cos\dfrac{\bar\theta}{N}\right)^{N/(N+1)} & \text{for } m^{N/(N+1)}\mu^{1/(N+1)}L \gg 1.
\end{cases}
\tag{8.16}
$$

9. General fermion masses

(1) $N = 2$

It is of interest to know the boson masses when the fermion masses are not degenerate. In the two-flavour case an analytic expression is obtained for $m_a \ll \mu$ in the $L \to \infty$ limit.

Start with (3.25), or more conveniently the truncated equation (6.3):

$$
\left\{ -\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} - \frac{2L}{\pi}\mathrm{e}^{-\pi/2\mu L} \sum_{a=1}^{2} m_a \bar{B}_a \cos\varphi_a \right\} f(\varphi) = \epsilon_0 f(\varphi).
\tag{9.1}
$$

Note that $\varphi_1 = \varphi$ and $\varphi_2 = \theta - \varphi$. In terms of $R_a = 8\pi m_a \bar{B}_a L^{-1} \mathrm{e}^{\pi/2\mu L} \langle\langle \cos\varphi_a \rangle\rangle_f$,

$$
\begin{pmatrix} \mu_1^2 \\ \mu_2^2 \end{pmatrix} = \begin{pmatrix} \mu^2 \\ 0 \end{pmatrix} + \frac{R_1 + R_2}{2}
$$

$$
\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} = B(\mu_1 L)^{1/2} B(\mu_2 L)^{1/2} \left[\frac{B(\mu_1 L)}{B(\mu_2 L)}\right]^{\pm(R_1 - R_2)/2\mu^2}.
\tag{9.2}
$$

In the $L \to \infty$ limit, $\bar{B}_a \propto L$ so that $\langle\langle \cos\varphi \rangle\rangle_f = \cos\varphi_{\min}$ where the potential term in (9.1) has a minimum at $\varphi_{\min}$. In general at the minimum of a function $g(\varphi) = -\alpha\cos\varphi - \beta\sin\varphi$, $\mathrm{e}^{\mathrm{i}\varphi} = (\alpha + \mathrm{i}\beta)/\sqrt{\alpha^2 + \beta^2}$. Hence,

$$
R_a = 8\pi \frac{(m_a \bar{B}_a)^2 + m_1 m_2 \tilde{B}_1 \tilde{B}_2 \cos\theta}{\sqrt{(m_1 \tilde{B}_1)^2 + (m_2 \tilde{B}_2)^2 + 2m_1 m_2 \tilde{B}_1 \tilde{B}_2 \cos\theta}}
\tag{9.3}
$$

where $\tilde{B}_a = \bar{B}_a/L$. Solving (9.2) and (9.3) to the leading order in m_a/μ, one finds

$$
R_a = 2\mathrm{e}^{4\gamma/3}\mu^{2/3} \frac{m_a^2 + m_1 m_2 \cos\theta}{(m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta)^{1/3}}.
\tag{9.4}
$$

Consequently,

$$
\mu_2 = \mathrm{e}^{2\gamma/3}\mu^{1/3}(m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta)^{1/3}
\tag{9.5}
$$

and

$$
\begin{pmatrix} \langle\bar\psi_1\psi_1\rangle'_\theta \\ \langle\bar\psi_2\psi_2\rangle'_\theta \end{pmatrix} = -\frac{\mathrm{e}^{4\gamma/3}\mu^{2/3}}{2\pi(m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta)^{1/3}} \begin{pmatrix} m_1 + m_2 \cos\theta \\ m_2 + m_1 \cos\theta \end{pmatrix}.
\tag{9.6}
$$

In the symmetric case $m_1 = m_2 = m$ (9.5) reduces to (8.14).

Observe that there is no singularity at $\theta = \pi$ in a generic case $m_1 \neq m_2$. The boson mass and chiral condensates are smooth functions of θ with a period 2π. The singularity appears only when $m_1 = m_2 \neq 0$ in the two-flavour case. In a special case $m_1 = 0$ but $m_2 \neq 0$ ($m_2 \ll \mu$), $(\langle\bar\psi_1\psi_1\rangle'_\theta, \langle\bar\psi_2\psi_2\rangle'_\theta) = -(2\pi)^{-1}(\mathrm{e}^{4\gamma}\mu^2 m_2)^{1/3}(\cos\theta, 1)$ in this approximation.

(2) $N = 3$

The three-flavour case mimics physics of four-dimensional QCD. Three fermions, which one may call 'up', 'down', and 'strange' quarks, have different masses; $m_1 \sim m_2 \ll m_3$. We would like to see how the asymmetry in masses affects chiral condensates and boson masses.

We concentrate on the $L \to \infty$ ($T = 0$) limit in the truncated theory. As in the $N = 2$ case one needs to find the location of the minimum of the potential

$$g(\varphi_1, \varphi_2; \theta) = \lim_{L \to \infty} \frac{\pi}{3L^2} V(\varphi) = -\sum_{a=1}^{3} m_a \tilde{B}_a \cos \varphi_a \tag{9.7}$$

where $\varphi_3 \equiv \theta - \varphi_1 - \varphi_2$. Here $\tilde{B}_a$'s are determined by the boson masses μ_α's and eigenvectors:

$$\tilde{B}_a = \frac{e^\gamma}{4\pi} \prod_{\alpha=1}^{3} \mu_\alpha^{(U_{\alpha a})^2}$$

$$\mathcal{K} = \frac{\mu^2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} R_1 & & \\ & R_2 & \\ & & R_3 \end{pmatrix} = U^t \begin{pmatrix} \mu_1^2 & & \\ & \mu_2^2 & \\ & & \mu_3^2 \end{pmatrix} U \tag{9.8}$$

$$R_a = 8\pi m_a \tilde{B}_a \cos \varphi_a^{\min}.$$

$\varphi_a^{\min}$ is the location of the minimum of $g(\varphi)$. The set of equations (9.7) and (9.8) must be solved simultaneously. Chiral condensates are given by

$$\langle \bar{\psi}_a \psi_a \rangle_\theta = -2\tilde{B}_a \cos \varphi_a^{\min} + \frac{e^{2\gamma}}{\pi} m_a. \tag{9.9}$$

When fermion masses are degenerate, $\varphi_a^{\min} = \frac{1}{3}\bar{\theta}$, as shown in section 8. At $\theta = \pi$ the location of the minimum changes discontinuously, which induces singular behaviour in physical quantities. We show that the singularity disappears if the asymmetry in fermion masses is sufficiently large. In [11] this problem was analysed by examining $g(\varphi)$, but without solving (9.8).

When $m_1 = m_2 < m_3$, $\varphi_1 = \varphi_2$ at the minimum of $g(\varphi)$. At $\theta = 0$, $\varphi_1 = \varphi_2 = 0$. As θ increases, $\varphi_1 = \varphi_2$ also increases. At $\theta = \pi$ it reaches $\varphi_1 = \varphi_2 = \varphi_c$ whose value depends on m_a's. As m_3 gets bigger and bigger with $m_1 = m_2$ kept fixed, φ_c approaches $\frac{1}{2}\pi$. For instance $\varphi_c = (0.467, 0.486, 0.499)\pi$ for $m_1/\mu = 0.01$ and $m_3/\mu = (0.02, 0.03, 0.1)$. As θ exceeds π, the minimum jumps to $\varphi_1 = \varphi_2 = -\varphi_c$ and returns to $\varphi_1 = \varphi_2 = 0$ at $\theta = 2\pi$. The singular behaviour at $\theta = \pi$ remains. This is expected as $m_3 \gg m_1 = m_2$ corresponds to the two-flavour case.

Now we add a small asymmetry in the light fermions. Figure 5 depicts the location of the minimum of $g(\varphi)$ as θ varies from $-\pi$ to π. A small asymmetry in m_1 and m_2, does not change the behaviour near $\theta = 0$, but significantly affects the behaviour near $\theta = \pm\pi$. At $(m_1, m_2, m_3)/\mu = (0.01, 0.011, 0.1)$ the minimum at $\theta = \pi$ is very close, but not quite equal, to $(\pi, 0)$. At $(m_1, m_2, m_3)/\mu = (0.01, 0.02, 0.1)$ the minimum at $\theta = \pi$ is located at $(\pi, 0)$, and the singularity in physical quantities at $\theta = \pi$ disappears.

In figure 6 we have plotted boson masses μ_1, μ_2, and μ_3 as functions of θ with given fermion masses. They correspond to $m_{\eta'}$, m_η, m_π in QCD. For $m_1 = m_2 = m_3 \ll \mu$, $\mu_1 \gg \mu_2 = \mu_3$. When $m_1 = m_2 < m_3 \ll \mu$, all μ_α's are different. The θ-dependence of each μ_α has similar behaviour. μ_1/μ, μ_2/μ, and μ_3/μ vary by 0.003, 0.007, and 0.1 in magnitude. The mass of the lightest boson, m_π, has the most θ-dependence.
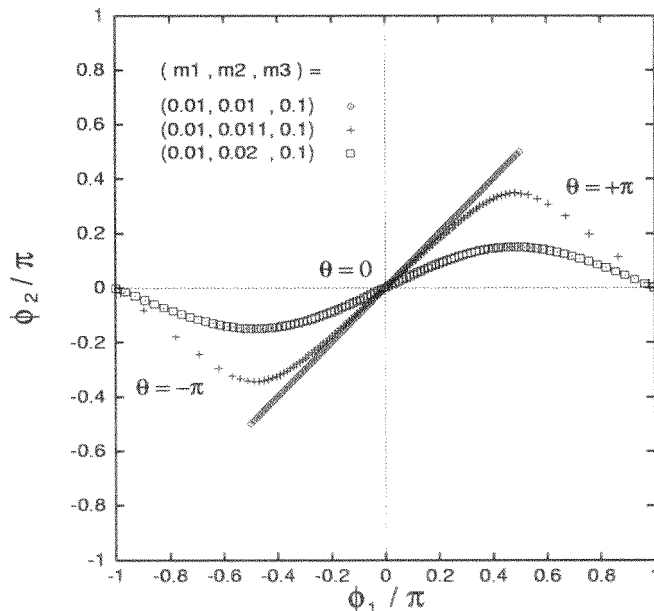
Figure 5. The location of the global minimum of the potential (9.7) in the $N = 3$ model with general fermion masses. As θ changes from $-\pi$ to π, the minimum in the figure moves from left to right. The values for the fermion masses, m_a's, are in a unit of μ. The location of the minimum discontinuously jumps at $\theta = \pm\pi$ for $(m_1, m_2, m_3)/\mu = (0.01, 0.01, 0.1)$ or $(0.01, 0.011, 0.1)$, but makes a continuous loop for $(0.01, 0.02, 0.1)$.
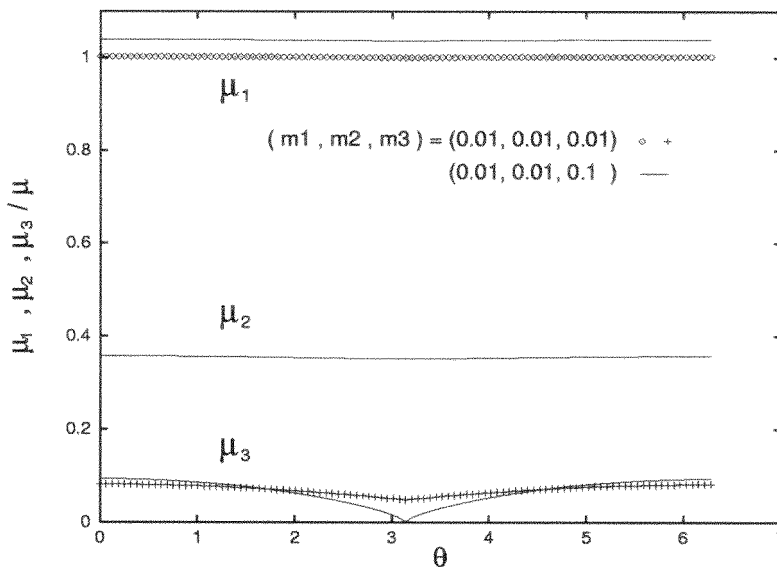

Figure 6. Boson masses in the $N = 3$ model at $T = 0$ with degenerate fermions $m_1 = m_2 = m_3$ and with $m_1 = m_2 < m_3$. In the former case $\mu_2 = \mu_3$. Fermion masses are in a unit of μ.

In figure 7 the θ-dependence of the mass of the lightest boson is plotted for various values of fermion masses. The cusp at $\theta = \pi$ persists as long as $m_1 = m_2$, but a small
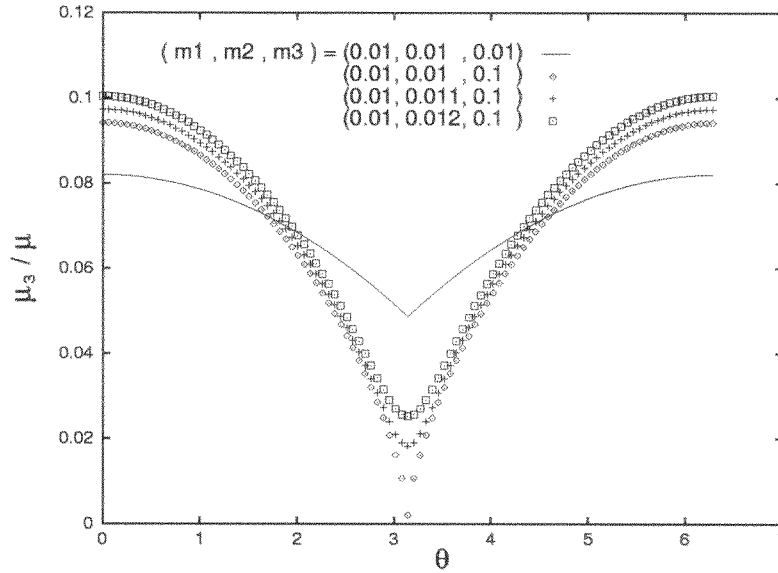
Figure 7. The mass, μ_3, of the lightest boson in the $N = 3$ model at $T = 0$ for various values of fermion masses. Fermion masses are in a unit of μ. The cusp at $\theta = \pi$ disappears as a small asymmetry in the two light fermions is added.

asymmetry in m_1 and m_2 changes it to smooth dependence. The mass μ_3 at $\theta = 0$ increases as the third fermion gets heavier as expected, whereas it decreases at $\theta = \pi$.

The θ-dependence of the location of the minimum of $g(\varphi)$ and the value of the mass μ_3 of the lightest boson induces nontrivial θ-dependence in the chiral condensates $\langle \bar{\psi}_a \psi_a \rangle_\theta$. In figure 8 a chiral condensates for $(m_1, m_2, m_3)/\mu = (0.01, 0.01, 0.01)$ and $(0.01,0.01,0.1)$ are depicted. In both cases there appear cusps at $\theta = \pi$. Notice that the magnitude of the condensates at $\theta = 0$ is insensitive to fermion masses. This is true only if the 'free theory background', $\langle \bar{\psi}\psi \rangle'_{\text{free}}$ is subtracted in the definition of the condensate (9.9).

There appears, however, a big difference in the θ-dependence of the condensates. When $(m_1, m_2, m_3)/\mu = (0.01, 0.01, 0.1)$, the third fermion ψ_3 is much heavier than the other two. The condensate $\langle \bar{\psi}_3 \psi_3 \rangle_\theta$ is more or less independent of θ, which is expected as the vacuum structure is mainly determined by light fermions.

When a small asymmetry in light fermions is added, condensates suffer a big change. See figure 9. With $(m_1, m_2, m_3)/\mu = (0.01, 0.011, 0.1)$ the θ-dependence in $\langle \bar{\psi}_1 \psi_1 \rangle_\theta$ is enhanced, whereas $\langle \bar{\psi}_2 \psi_2 \rangle_\theta$ develops a dip near $\theta = \pi$. A small asymmetry in m_1 and m_2 induces a big difference in $\langle \bar{\psi}_1 \psi_1 \rangle_\theta$ and $\langle \bar{\psi}_2 \psi_2 \rangle_\theta$ near $\theta = \pi$. The nonlinearity in equation (9.7) and (9.8) gives rise to such sensitive dependence.

It is interesting to recognize the similarity between the potential $g(\varphi)$ in (9.7) and the effective chiral Lagrangian proposed by Witten to describe low-energy behaviour of four-dimensional QCD [39]. In Witten's approach

$$V^{\text{Witten}}(U) = f_\pi^2 \left\{ -\frac{1}{2} \operatorname{Tr} M(U + U^\dagger) + \frac{k}{2N_c} (-i \ln \det U - \theta)^2 \right\} \qquad (9.10)$$

where U is the pseudoscalar field matrix and $M = \operatorname{diag}(m_{\text{u}}, m_{\text{d}}, m_{\text{s}})$ is the quark mass matrix. The second term represents contributions from instantons. k is O(1) in the large N_c (colour) limit.
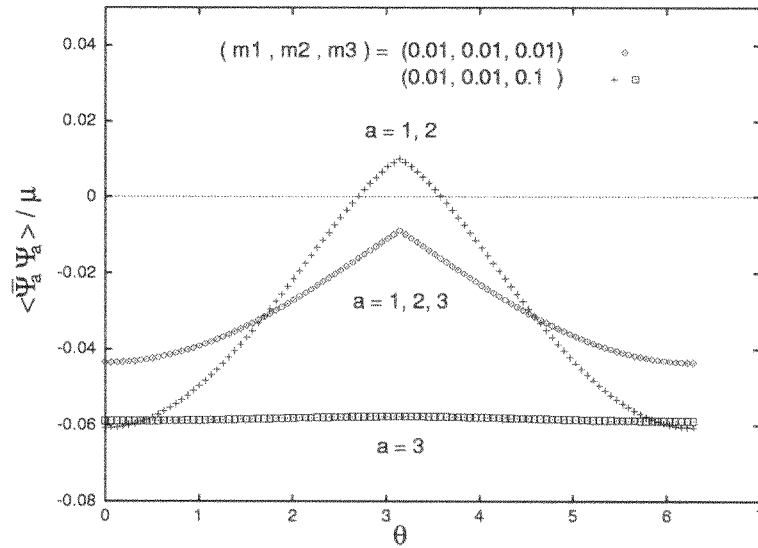
Figure 8. The θ-dependence of the chiral condensates in the $N = 3$ model at $T = 0$. Two cases, $(m_1, m_2, m_3)/\mu = (0.01, 0.01, 0.01)$ and $(0.01, 0.01, 0.1)$, are displayed. In the latter case the condensate of the heavy fermion has little dependence on θ, whereas the condensates of the light fermions show large dependence.
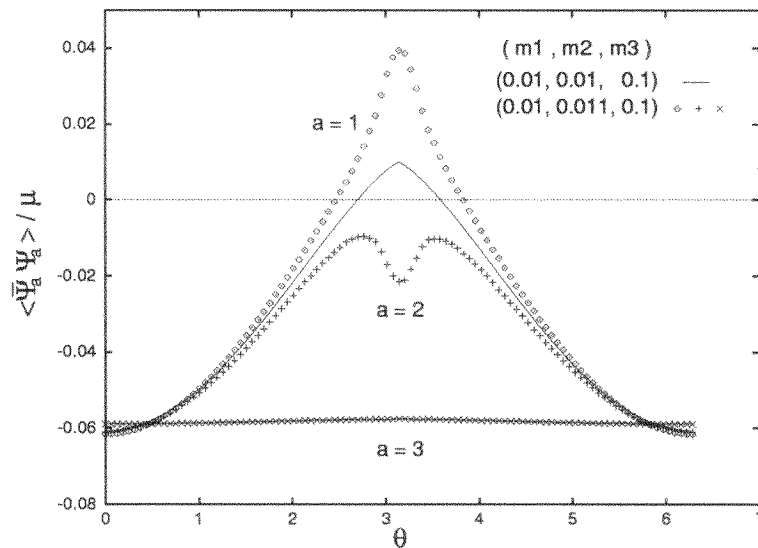

Figure 9. The θ-dependence of the chiral condensates in the $N = 3$ model at $T = 0$. Small asymmetry is added to masses of light fermions. The effect is minor near $\theta = 0$, but the condensates of the light fermions near $\theta = \pi$ are significantly affected.

The fact that $m_{\eta'}^2 \gg m_\pi^2, m_K^2, m_\eta^2$ implies that $k/N_c \gg m_a$. Diagonalize U and denote it by $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \mathrm{e}^{\mathrm{i}\phi_3})$. As the second term dominates over the first, $\sum \phi_a = \theta$ to the first approximation. With ϕ_3 eliminated, $V^{\mathrm{Witten}}(U)$ takes the same form as $g(\varphi)$ in (9.7). Consequently both models show qualitatively similar behaviour. Indeed, Witten has argued

that a small asymmetry in m_u and m_d, in addition to large asymmetry $m_\text{s} \gg m_\text{u}, m_\text{d}$ removes the singularity of physical quantities in θ at π, which is exactly what we are observing in the $N = 3$ Schwinger model. However, it should be noted that the coefficients in the potential $g(\varphi)$ have extra fermion mass-dependence coming from the factors $\tilde{B}_a$, which have significant effects near $\theta = \pi$.

10. Summary

In this paper the massive N-flavour Schwinger model was analysed in the generalized Hartree–Fock approximation. Dynamics of the zero-modes is determined by the Schrödinger equation for N degrees of freedom. The potential term in the Schrödinger equation depends on the boson spectrum in the oscillatory modes. The boson mass spectrum in turn depends on the ground state in the zero-mode sector. The ground state of the two sectors must be determined self-consistently.

   We have evaluated the boson spectrum and chiral condensates in the $N = 1, 2, 3$ models. In the $N = 1$ model we have found anomalous dependence of physical quantities on the fermion mass near $\theta = \pi$ at low temperature. In the $N = 3$ model physics near $\theta = \pi$ is very sensitive to the small asymmetry in fermion masses. Chiral dynamics in the $N = 3$ model resembles with that in QCD in four dimensions.

Acknowledgments

Appendix A. Correlation functions

Green's function for a scalar field with a mass μ on S^1, excluding the zero mode, is

$$\Delta(x; \mu, L) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{(2\pi n)^2 + (\mu L)^2}} \cos \frac{2\pi \mathrm{i} n x}{L}. \tag{A.1}$$

In the massless case ($\mu = 0$) it is given by

$$\Delta(x; 0, L) = -\frac{1}{4\pi} \ln 2 \left(1 - \cos \frac{2\pi x}{L}\right)$$

$$\mathrm{e}^{2\pi \Delta(x; 0, L)} \sim \frac{L}{2\pi x} \qquad \text{for } \left|\frac{2\pi x}{L}\right| \ll 1. \tag{A.2}$$

In terms of

$$I[s; a, b] = \sum_{n=1}^{\infty} \frac{\cos 2\pi n b}{(n^2 + a^2)^s} \tag{A.3}$$

$2\pi \Delta(x; \mu, L) = I[\frac{1}{2}; \mu L/2\pi, x/L]$ and $B(\mu L) = \exp\{-I[\frac{1}{2}; \mu L/2\pi, 0] + I[\frac{1}{2}; 0, 0]\}$. For $|b| < 1$

$$I[s; a, b] = \int_0^{\infty} \mathrm{d}t \, \frac{\cos 2\pi b t}{(t^2 + a^2)^s} - \frac{1}{2a^{2s}} + 2 \sin s\pi \int_a^{\infty} \mathrm{d}t \, \frac{\cosh 2\pi b t}{(t^2 - a^2)^s (\mathrm{e}^{2\pi t} - 1)}$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)} \left|\frac{\pi b}{a}\right|^{s - \frac{1}{2}} K_{s - \frac{1}{2}}(2\pi |ab|) - \frac{1}{2a^{2s}} + 2 \sin s\pi \int_a^{\infty} \mathrm{d}t \, \frac{\cosh 2\pi b t}{(t^2 - a^2)^s (\mathrm{e}^{2\pi t} - 1)}.$$

$$(A.4)$$

The above formula is valid for an arbitrary s by analytic continuation.

It then follows that

$$2\pi \Delta(x; \mu, L) = K_0(|\mu x|) - \frac{\pi}{\mu L} + 2 \int_1^\infty du \, \frac{\cosh \mu x u}{(e^{\mu L u} - 1)\sqrt{u^2 - 1}}. \qquad (A.5)$$

Recalling that $K_0(z) \sim -\ln z - \gamma + \ln 2$ for $z \ll 1$ and $\sim \sqrt{\frac{\pi}{2z}}e^{-z}$ for $z \gg 1$, and noticing for $\mu L \gg 1$

$$2 \int_1^\infty du \, \frac{\cosh \mu x u}{(e^{\mu L u} - 1)\sqrt{u^2 - 1}} \sim \sqrt{\frac{\pi}{2\mu(L - x)}}e^{-\mu(L-x)} + \sqrt{\frac{\pi}{2\mu(L + x)}}e^{-\mu(L+x)} \qquad (A.6)$$

one finds

$$e^{2\pi \Delta(x; \mu, L)} \sim \begin{cases} 1 & \text{for } \mu L \gg 1 \quad \frac{x}{L} \ll 1 \quad \mu x \gg 1 \\ \dfrac{2e^{-\gamma}}{\mu x} & \text{for } \mu L \gg 1 \quad \frac{x}{L} \ll 1 \quad \mu x \ll 1. \end{cases} \qquad (A.7)$$

Appendix B. Normal ordering and Bogoliubov transformation

In the subsequent discussions, we make frequent use of identities: $(1) : e^A := e^{A^-}e^{A^+}$, (2) $e^A e^B = e^{\frac{1}{2}[A,B]}e^{A+B} = e^{[A,B]}e^B e^A$, and $(3) : e^A :: e^B := e^{[A^+, B^-]} : e^{A+B}$: where A^+ and A^- denote the annihilation and creation operator parts of A, respectively. With massless fields (2.2)

$$[\phi_\pm^a(t, x)^{(+)}, \phi_\pm^b(0, 0)^{(-)}] = -\delta^{ab} \frac{1}{4\pi} \ln\{1 - e^{-2\pi i(t \pm x - i\epsilon)/L}\}. \qquad (B.1)$$

We also note that

$$\frac{1}{L} \left\{ \frac{e^{+i\pi x/L}}{1 - e^{+2\pi i(x+i\epsilon)/L}} + \frac{e^{-i\pi x/L}}{1 - e^{-2\pi i(x-i\epsilon)/L}} \right\} = e^{i\pi x/L} \delta_L(x)$$

$$\frac{e^{+i\pi x/L}}{1 - e^{+2\pi i(x+i\epsilon)/L}} = \frac{-1}{2\pi i} \left(\frac{L}{x + i\epsilon} + \frac{\pi^2}{6} \frac{x}{L} + \frac{7\pi^4}{360} \frac{x^3}{L^3} + \cdots \right). \qquad (B.2)$$

We have seen that boson fields become massive due to the Coulomb interaction and fermion masses. When this happens, the vacuum also changes and in two dimensions nonvanishing chiral condensates result.

It is most convenient to work in the Schrödinger picture. A boson field is generically denoted by $\phi(x)$ with its conjugate $\Pi(x)$. On a circle they are expanded as

$$\phi(x) = \sum_{n \neq 0} \frac{1}{\sqrt{2\omega_n(\mu)L}} \{c_n(\mu)e^{ip_n x} + c_n^\dagger(\mu)e^{-ip_n x}\}$$

$$\Pi(x) = -i \sum_{n \neq 0} \sqrt{\frac{\omega_n(\mu)}{2L}} \{c_n(\mu)e^{ip_n x} - c_n^\dagger(\mu)e^{-ip_n x}\} \qquad (B.3)$$

where $p_n = 2\pi n/L$ and $\omega_n(\mu) = \sqrt{p_n^2 + \mu^2}$. Annihilation and creation operators $c_n(\mu)$ and $c_n^\dagger(\mu)$ are defined with respect to a mass μ. The left- and right-moving modes, $\phi_+(x)$ and $\phi_-(x)$, of $\phi = \phi_+ + \phi_-$ are defined by the $n < 0$ and $n > 0$ components in (B.3). In the massless limit this corresponds to (2.2) in the Schrödinger picture.

Sometimes we need to treat $\phi_\pm$'s separately. One finds

$$\phi_\pm(x) = \frac{1}{2}\phi(x) \pm \frac{1}{2L}\int_0^L dy\, F(x-y;\mu)\Pi_\phi(y)$$

$$F(x;\mu) = \sum_{n=1}^\infty \frac{i}{\omega_n(\mu)}\{e^{-2\pi inx/L} - e^{+2\pi inx/L}\}. \tag{B.4}$$

Note that the definition of $\phi_\pm(x)$ depends on the reference mass μ as opposed to that of $\phi(x)$ and $\Pi_\phi(x)$. In the massless theory $F'(x;0) = L\delta_L(x) - 1$ so that $\Pi(x) = \phi'_+(x) - \phi'_-(x)$.

We have an identity among $c_n(\mu)$'s with different μ's:

$$c_n(\mu_1) = \cosh\theta_n(\mu_1;\mu_2)c_n(\mu_2) + \sinh\theta_n(\mu_1;\mu_2)c^\dagger_{-n}(\mu_2)$$

$$\begin{bmatrix} \cosh\theta_n(\mu_1;\mu_2) \\ \sinh\theta_n(\mu_1;\mu_2) \end{bmatrix} = \frac{1}{2}\left\{ \sqrt{\frac{\omega_n(\mu_1)}{\omega_n(\mu_2)}} \pm \sqrt{\frac{\omega_n(\mu_2)}{\omega_n(\mu_1)}} \right\}. \tag{B.5}$$

In other words the change in the boson mass induces a Bogoliubov transformation. The vacuum with respect to a boson mass μ is defined by $c_n(\mu)|\text{vac};\mu\rangle = 0$.

In our formalism fermion fields are first bosonized in the interaction picture defined by massless bosons. Boson fields then acquire masses and the vacuum is redefined. In particular boson fields are normal-ordered with respect to physical boson masses. One useful relation is

$$e^{i\alpha\phi(x)} = \exp\left\{ -\frac{\alpha^2}{2L}\sum_{n=1}^\infty \frac{1}{\omega_n(\mu)} \right\} N_\mu[e^{i\alpha\phi(x)}] \tag{B.6}$$

from which it follows [3, 30]

$$N_0[e^{i\alpha\phi(x)}] = B(\mu L)^{\alpha^2/4\pi} N_\mu[e^{i\alpha\phi(x)}]$$

$$B(\mu L) = \exp\left\{ -\sum_{n=1}^\infty \left(\frac{1}{\sqrt{n^2 + (\mu L/2\pi)^2}} - \frac{1}{n} \right) \right\}$$

$$= \frac{\mu L}{4\pi} \exp\left\{ \gamma + \frac{\pi}{\mu L} - 2\int_0^\infty \frac{dx}{e^{\mu L\cosh x} - 1} \right\}. \tag{B.7}$$

In the following we make use of simplified notation: $\phi_\pm(x)$ refers to the massless field, $\omega_n = \omega_n(0)$ and $\theta_n = \theta_n(0;\mu)$. $N_\mu[A(c,c^\dagger)]$ denotes that the operator A is normal-ordered with respect to $c_n(\mu)$ and $c^\dagger_n(\mu)$. Some useful identities are

$$N_0[e^{i\alpha\phi(x)+i\beta\phi(y)}]/N_\mu[e^{i\alpha\phi(x)+i\beta\phi(y)}] = B(\mu L)^{(\alpha^2+\beta^2)/4\pi}e^{-\alpha\beta\{\Delta(x-y;\mu,L)-\Delta(x-y;0,L)\}} \tag{B.8}$$

$$N_0[e^{i\alpha_+\phi_+(x)+i\alpha_-\phi_-(x)}]/N_\mu[e^{i\alpha_+\hat\phi_+(x)+i\alpha_-\hat\phi_-(x)}]$$

$$= \exp\left\{ -\sum_{n=1}^\infty \frac{1}{2\omega_n L}[2\alpha_+\alpha_-\cosh\theta_n\sinh\theta_n + (\alpha_+^2 + \alpha_-^2)\sinh^2\theta_n] \right\}$$

$$\hat\phi_\pm(x;\mu) = \sum_{n=1}^\infty \frac{1}{\sqrt{2\omega_n L}}\left\{ \begin{pmatrix} \sinh\theta_n \\ \cosh\theta_n \end{pmatrix}(c_n(\mu)e^{ip_nx} + c^\dagger_n(\mu)e^{-ip_nx}) \right.$$

$$\left. + \begin{pmatrix} \cosh\theta_n \\ \sinh\theta_n \end{pmatrix}(c_{-n}(\mu)e^{-ip_nx} + c^\dagger_{-n}(\mu)e^{ip_nx}) \right\} \tag{B.9}$$

$$N_0[e^{i\alpha\phi_+(x)-i\alpha\phi_-(x)+i\beta\phi_+(y)-i\beta\phi_-(y)}]/N_\mu[e^{i\alpha\hat\chi_+(x)-i\alpha\hat\chi_-(x)+i\beta\hat\chi_+(y)-i\beta\hat\chi_-(y)}]$$

$$= B(\mu L)^{(\alpha^2+\beta^2)/4\pi}e^{-\alpha\beta\{\Delta(x-y;\mu,L)-\Delta(x-y;0,L)\}}e^{-\frac{1}{2}(\alpha^2+\beta^2)h(0;\mu,L)-\alpha\beta h(x-y;\mu,L)}$$

$$\hat{\chi}_{\pm}(x) = \sum_{n=1}^{\infty} \frac{1}{\omega_n} \sqrt{\frac{\omega_n(\mu)}{2L}} (c_{\mp n}(\mu) e^{\mp i p_n x} + \text{h.c.})$$

$$h(x; \mu, L) = \frac{1}{2L} \sum_{n \neq 0} \frac{\mu^2}{\omega_n^2 \omega_n(\mu)} e^{i p_n x}. \tag{B.10}$$

Appendix C. Useful identities for zero modes

(1) Free massless fermion

In a theory of a free massless fermion the Hamiltonian in the bosonization method is given by

$$H = \frac{\pi}{2L} (4p^2 + \tilde{p}^2) + \int dx \, \frac{1}{2} (\dot{\phi}^2 + \phi'^2) \tag{C.1}$$

where $\phi = \phi_+ + \phi_-$ represents a massless boson defined in (2.2). The vacuum state is $|\Psi_{\text{vac}}\rangle = |n = 0\rangle$, i.e. $p|\Psi_{\text{vac}}\rangle = 0$. Note that $\langle e^{i\alpha q}\rangle = \delta_{\alpha,0}$. Since $q(t) = q + 4\pi pt/L$,

$$\langle e^{iaq(t)} e^{ibq(0)}\rangle = \delta_{a+b} e^{-2\pi i a^2 t/L}$$

$$\langle T[e^{iaq(t)} e^{ibq(0)}]\rangle = \delta_{a+b} e^{-2\pi i a^2 |t|/L}. \tag{C.2}$$

(2) QED$_2$

If all fermions are massless, the model is solved in the operator form in the Heisenberg picture. The solution to (3.4) in the N-flavour model is

$$\Theta'_W(t) = \Theta'_W \cos \mu t + \frac{\pi \mu L}{N} P_W \sin \mu t$$

$$P_W(t) = P_W \cos \mu t - \frac{N}{\pi \mu L} \Theta'_W \sin \mu t$$

$$q'_a(t) = q'_a + \frac{4\pi t}{L} \left(p_a - \frac{1}{N} \sum p_b \right) \tag{C.3}$$

$$p_a(t) = p_a$$

$$\tilde{q}_a(t) = \tilde{q}_a + \frac{\pi t}{L} \tilde{p}_a$$

$$\tilde{p}_a(t) = \tilde{p}_a.$$

Here Θ'_W and q'_a are defined in (3.3).

In the $N = 1$ model, for an integer ℓ,

$$\langle e^{i\ell q + i\beta(\Theta_W + 2\pi p) + i\gamma p_W}\rangle = e^{i\ell\theta} e^{-\pi \mu L \beta^2/4 - (\gamma - 2\pi\ell)^2/4\pi \mu L}. \tag{C.4}$$

It follows from (C.3) and (C.4) that, for an integer $\alpha + \beta$,

$$\langle e^{i\alpha q(t)} e^{i\beta q(0)}\rangle = \exp\left\{ i(\alpha + \beta)\theta - \frac{\pi(\alpha^2 + \beta^2)}{\mu L} - \frac{2\pi \alpha \beta}{\mu L} e^{-i\mu t} \right\}. \tag{C.5}$$

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